14. ‡ **Heat conduction.** (p.89) Consider an unknown one-dimensional heat source, $g(x)$ [W/m], distributed along x between $+1$ and -1. The temperature distribution is $T(x)$ degrees K. The heat-source density function, $g(x)$, is uncertain and belongs to an info-gap model.

The differential equation for heat conduction is:

$$0 = \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} + \frac{g(x)}{k} \tag{43}$$

where k is the thermal conductivity, in units of W·m/K.

Safe operation requires that the central temperature be less than a critical value:

$$T(0) \le T_{\mathrm{c}} \tag{44}$$

We are able to control the surface temperatures, $T(\pm 1)$.

Consider the following two info-gap models.

Uniform bound:

$$\mathcal{U}(h, \widetilde{g}) = \{ g(x) : \ |g(x) - \widetilde{g}| \le h \}, \quad h \ge 0 \tag{45}$$

Fourier ellipsoid bound:

$$\mathcal{U}(h, \widetilde{g}) = \left\{ g(x) = \widetilde{g} + c^T \gamma(x) : \ c^T W c \le h^2 \right\}, \quad h \ge 0 \tag{46}$$

where W is a known, real, symmetric, positive definite matrix and $\gamma(x)$ is the vector:

$$\gamma(x) = (\cos \pi x, \, \cos 2\pi x, \, \ldots, \, \cos N\pi x)^T \tag{47}$$

(a) Study the robustness and the opportuneness as a function of surface temperature, for each of the above info-gap models of heat-source uncertainty. Discuss the meaning of these two immunity functions. Develop general expressions for the immunity functions and then consider the special case where W is the following diagonal matrix:

$$W = \mathrm{diag}\left(\frac{1}{n^2}, \ n = 1, \ldots, 6 \right) \tag{48}$$

(b) Now consider a specific numerical case. The material is steel, whose thermal conductivity is $k = 17.3$ [W·m/K]. The critical temperature is $T_{\mathrm{c}} = 400$ [K]. The nominal heat-source density is $\widetilde{g} = 250$ [W/m]. For each of the info-gap models, what range of surface temperature values are very reliable? Very unreliable? Compare the results for the two info-gap models.

Solution for problem 14. (p.10)

Our procedure:

1. Solve eq.(43) to express the temperature at $x = 0$ as a function of the unknown heat source term.

2. Derive robustness functions.

3. Derive opportuneness functions.

Part 1 of solution to problem 14.

We must first **solve eq.(43) to express the temperature at $x = 0$ as a function of the unknown heat source term.** This is the mechanical model of the system. We express the differential equation as:

$$\ddot{T}(x) = -\frac{1}{k}g(x) \tag{564}$$

In terms of differentials this can be written:

$$\mathrm{d}(\dot{T}) = -\frac{1}{k}g(x)\mathrm{d}x \tag{565}$$

which can be integrated indefinitely as:

$$\int \mathrm{d}(\dot{T}) = -\frac{1}{k}\int g(x)\,\mathrm{d}x \tag{566}$$

which yields:

$$\dot{T}(y) = -\frac{1}{k}\int g(x)\,\mathrm{d}x + c_1 \tag{567}$$

Integrating indefinitely again:

$$\int \mathrm{d}T(y) = -\frac{1}{k}\int\int g(x)\,\mathrm{d}x\,\mathrm{d}y + \int c_1\,\mathrm{d}y \tag{568}$$

one obtains:

$$T(y) = \underbrace{-\frac{1}{k}\int\int g(x)\,\mathrm{d}x\,\mathrm{d}y}_{A(y)} + c_1 y + c_2 \tag{569}$$

where $A(y)$ depends on the unknown power distribution $g(x)$.

Evaluate this relation at the two controlled boundardies, $y = 1$ and $y = -1$:

$$T(1) = A(1) + c_1 + c_2 \tag{570}$$

$$T(-1) = A(-1) - c_1 + c_2 \tag{571}$$

Solving for the integration constants:

$$c_2 = \frac{T(1) + T(-1) - A(1) - A(-1)}{2} \tag{572}$$

$$c_1 = \frac{T(1) - T(-1) - A(1) + A(-1)}{2} \tag{573}$$

Evaluate eq.(569) at $y = 0$:

$$T(0) = A(0) + c_2 \tag{574}$$

$$= A(0) + \underbrace{\frac{T(1) + T(-1)}{2}}_{\overline{T}} - \frac{1}{2}A(1) - \frac{1}{2}A(-1) \tag{575}$$

where $\overline{T}$ is the mean control temperature.

For example, if $g(x) = g_0 =$ constant one finds:

$$A(y) = -\frac{1}{k}\int\int g_0 \,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{k}\int g_0 y\,\mathrm{d}y = -\frac{g_0 y^2}{2k} \tag{576}$$

In this case, with $g(x) = g_0 =$ constant, eq.(575) becomes:

$$T(0) = \overline{T} + \frac{g_0 1^2}{4k} + \frac{g_0(-1)^2}{4k} \tag{577}$$

$$= \overline{T} + \frac{g_0}{2k} \tag{578}$$

Part 2 of solution to problem 14: Robustness function.

The robustness is the greatest horizon of uncertainty at which failure cannot occur:

$$\widehat{h} = \max\left\{h : \left(\max_{g\in\mathcal{U}(h,\widetilde{g})} T(0)\right) \le t_{\mathrm{c}}\right\} \tag{579}$$

Let us denote the inner maximum by $\mu(h)$, which is the inverse of the robustness function.

(a) First consider the uniform-bound info-gap model of eq.(45). From physical reasoning we see that $\mu(h)$ occurs for $g(x) = \widetilde{g} + h$. From eq.(578) we find:

$$\mu(h) = \overline{T} + \frac{\widetilde{g} + h}{2k} \tag{580}$$

Equating to t_{c} and solving for h yields the robustness:

$$\overline{T} + \frac{\widetilde{g} + h}{2k} = t_{\mathrm{c}} \quad\Longrightarrow\quad \widehat{h}_1 = 2k\left(t_{\mathrm{c}} - \overline{T}\right) - \widetilde{g} \tag{581}$$

or zero if this is negative.

(b) Now consider the Fourier-bound info-gap model for uncertainty in the distribution of the heat source.

We begin by developing a more convenient expression for $A(y)$ in eq.(569):

$$A(y) = -\frac{1}{k}\int\int g(x)\,\mathrm{d}x\,\mathrm{d}y \tag{582}$$

$$= -\frac{1}{k}\int\int [\widetilde{g} + c^T\gamma(x)]\,\mathrm{d}x\,\mathrm{d}y \tag{583}$$

$$= \underbrace{-\frac{\widetilde{g}}{k}\int\int \mathrm{d}x\,\mathrm{d}y}_{I_1} - \frac{1}{k}\underbrace{\int\int c^T\gamma(x)\,\mathrm{d}x\,\mathrm{d}y}_{I_2} \tag{584}$$

which defines two integrals, I_1 and I_2. We easily find:

$$I_1 = -\frac{\widetilde{g}}{k}\int y\,\mathrm{d}y = -\frac{\widetilde{g}y^2}{2k} \tag{585}$$

The second integral is:

$$I_2 = \int\int c^T\gamma(x)\,\mathrm{d}x\,\mathrm{d}y \tag{586}$$

$$= \sum_{n=1}^{N} c_n \int\int \cos n\pi x\,\mathrm{d}x\,\mathrm{d}y \tag{587}$$

$$= \sum_{n=1}^{N} c_n \frac{-\cos n\pi y}{n^2\pi^2} \tag{588}$$

$$= c^T\zeta(y) \tag{589}$$

where $\zeta(y)$ is a vector whose elements are defined in eq.(588). Note that $\zeta(1) = \zeta(-1)$ because cosine is a symmetric function.

We can now re-write eq.(584) as:

$$A(y) = I_1 - \frac{1}{k} I_2 = -\frac{\widetilde{g} y^2}{2k} - \frac{1}{k} c^T \zeta(y) \tag{590}$$

We can now derive an explicit expression for the inner temperature, eq.(575):

$$\begin{aligned} T(0) &= \overline{T} + A(0) - \frac{1}{2} A(1) - \frac{1}{2} A(-1) & (591) \\ &= \overline{T} - \frac{1}{k} c^T \zeta(0) - \frac{1}{2}\left(-\frac{\widetilde{g}}{2k} - \frac{1}{k} c^T \zeta(1) \right) - \frac{1}{2}\left(-\frac{\widetilde{g}}{2k} - \frac{1}{k} c^T \zeta(-1) \right) & (592) \\ &= \overline{T} + \frac{\widetilde{g}}{2k} + \frac{1}{k} c^T \underbrace{[-\zeta(0) + 2\zeta(1)]}_{\eta} & (593) \\ &= \overline{T} + \frac{\widetilde{g}}{2k} + \frac{1}{k} c^T \eta & (594) \end{aligned}$$

η is a known vector. From the definition of $\zeta_n(y)$ in eq.(588), we find that the nth element of η is:

$$\eta_n = \frac{1}{n^2 \pi^2} (\cos 0 - 2 \cos n\pi) = \frac{1}{n^2 \pi^2} (1 - 2(-1)^n) \tag{595}$$

We are now prepared to seek the maximum internal temperature. The basic optimization to be performed is:

$$\max c^T \eta \quad \text{subject to the constraint} \quad c^T W c = h^2 \tag{596}$$

We use Lagrange optimization. Define:

$$H = c^T \eta + \lambda (h^2 - c^T W c) \tag{597}$$

The condition for an extremum is:

$$\frac{\partial H}{\partial c} = 0 = \eta - 2\lambda W c \tag{598}$$

which implies that an optimizing vector of Fourier coefficients is:

$$c = \frac{1}{2\lambda} W^{-1} \eta \tag{599}$$

The constraint is used to determine the Lagrange multiplier:

$$h^2 = \frac{1}{4\lambda^2} \eta^T W^{-1} \eta \tag{600}$$

which implies:

$$\frac{1}{2\lambda} = \frac{\pm h}{\sqrt{\eta^T W^{-1} \eta}} \tag{601}$$

Now the optimizing Fourier vector becomes:

$$c = \frac{\pm h}{\sqrt{\eta^T W^{-1} \eta}} W^{-1} \eta \tag{602}$$

Hence the desired maximum is:

$$\max_{g \in \mathcal{U}_2(h, \widetilde{g})} c^T \eta = h \sqrt{\eta^T W^{-1} \eta} \tag{603}$$

So, the maximum temperature at the midpoint, up to uncertainty h, is:

$$\mu(h) = \overline{T} + \frac{\widetilde{g}}{2k} + \frac{h}{k} \sqrt{\eta^T W^{-1} \eta} \tag{604}$$

The robustness is obtained by equating this maximum temperature to the critical temperature $t_{\rm c}$ and solving for the uncertainty parameter, resulting in:

$$\widehat{h}_2 = \frac{k\left(t_{\rm c} - \overline{T} - \dfrac{\widetilde{g}}{2k}\right)}{\sqrt{\eta^T W^{-1}\eta}} \tag{605}$$

unless this is negative, in which case the robustness is zero.

(c) In the special case of the diagonal matrix W of eq.(48) we can easily evaluate the denominator of eq.(605) to find:

$$\frac{1}{\sqrt{\eta^T W^{-1}\eta}} \approx 3.0172 \tag{606}$$

The robustness for the Fourier-ellipsoid uncertainty model is:

$$\widehat{h}_2 = 3.0k\left(t_{\rm c} - \overline{T}\right) - 1.5\widetilde{g} \tag{607}$$

Both robustness functions have the same units as $\widetilde{g}$, the nominal power density. We can calibrate both robustnesses with respect to $\widetilde{g}$ as:

$$\frac{\widehat{h}_1}{\widetilde{g}} = \frac{2k\left(t_{\rm c} - \overline{T}\right)}{\widetilde{g}} - 1 \tag{608}$$

$$\frac{\widehat{h}_2}{\widetilde{g}} = \frac{3.0k\left(t_{\rm c} - \overline{T}\right)}{\widetilde{g}} - 1.5 \tag{609}$$

In either case, if $\widehat{h}_n/\widetilde{g} \gg 1$ then the system is immune to power-density fluctuations much greater than the nominal power density, which suggests a high level of reliability. On the other hand, if $\widehat{h}_n/\widetilde{g} \ll 1$ then power-density fluctuations which are much smaller than the nominal density entail the possibility of failure so the system is unreliable:

$$\frac{\widehat{h}_n}{\widetilde{g}} \gg 1 \implies \text{high reliability} \tag{610}$$

$$\frac{\widehat{h}_n}{\widetilde{g}} \ll 1 \implies \text{low reliability} \tag{611}$$

We are now in a position to answer the questions: what value of the mean control temperature, $\overline{T}$, insures high reliability, and what value entails high risk?

Let us adopt a value of $\widehat{h}_n/\widetilde{g} = 3$ as a large value implying high reliability, and a value of $\widehat{h}_n/\widetilde{g} = 1/3$ as a small value entailing low reliability.

For the uniform-bound info-gap model we find:

$$\frac{\widehat{h}_1}{\widetilde{g}} \geq 3 \implies \overline{T} \leq 371.1\,{\rm K} \tag{612}$$

$$\frac{\widehat{h}_1}{\widetilde{g}} \leq \frac{1}{3} \implies \overline{T} \geq 390.4\,{\rm K} \tag{613}$$

This means that, for the uniform-bound uncertainty model $\mathcal{U}_1(h)$, a control temperature of 371.1 K (or lower) assures high reliability while 390.4 K (or higher) entails substantial risk.

Performing the same calculations for the Fourier-ellipsoid model we find:

$$\frac{\widehat{h}_2}{\widetilde{g}} \geq 3 \implies \overline{T} \leq 378.3\,{\rm K} \tag{614}$$

$$\frac{\widehat{h}_2}{\widetilde{g}} \leq \frac{1}{3} \implies \overline{T} \geq 391.2\,{\rm K} \tag{615}$$

Figure 22: Reliable ranges of $\overline{T}$ for problem 14.

This means that, for the Fourier ellipsoid-bound uncertainty model $\mathcal{U}_2(h)$, a control temperature of 378.3 K (or lower) assures high reliability while 391.2 K (or higher) entails substantial risk.

We see that the mean control temperatures for the Fourier-ellipsoid info-gap model are shifted to higher values than for the uniform-bound model.

Part 3 of solution to problem 14: Opportuneness function.

The opportuneness is the least value of the horizon of uncertainty at which the midline temperature can be (though need not be) far below failure:

$$\widehat{\beta} = \min\left\{ h : \left(\min_{g\in\mathcal{U}(h,\widetilde{g})} T(0) \right) \le t_{\mathrm{w}} \right\} \tag{616}$$

where

$$t_{\mathrm{w}} \ll t_{\mathrm{c}} \tag{617}$$

(a) First consider the uniform bound. The lowest possible internal temperature, up to uncertainty h, is:

$$\min_{g\in\mathcal{U}_1(h,\widetilde{g})} T(0) = \overline{T} + \frac{\widetilde{g} - h}{2k} \tag{618}$$

Equating the minimum internal temperature to the windfall value t_{w} and solving for h yields the opportunity:

$$\overline{T} + \frac{\widetilde{g} - h}{2k} = t_{\mathrm{w}} \quad \Longrightarrow \quad \widehat{\beta}_1 = 2k\left(\overline{T} - t_{\mathrm{w}}\right) + \widetilde{g} \tag{619}$$

unless this expression is negative, in which case the opportuneness function takes the value zero.

Note that this increases with increasing windfall temperature t_{w}, unlike $\widehat{h}$ which decreases with increasing critical temperature t_{c}.

Comparing this opportuneness function with the robustness function of eq.(581) on p.90, we see that these immunity functions are **sympathetic:** a change in the control temperature which improves one, also improves the other.

(b) Now consider the Fourier ellipsoid bound. Arguing as in eq.(604), the lowest possible internal temperature, up to uncertainty h, is:

$$\min_{g\in\mathcal{U}_2(h,\widetilde{g})} T(0) = \overline{T} + \frac{\widetilde{g}}{2k} - \frac{h}{k}\sqrt{\eta^T W^{-1}\eta} \tag{620}$$

The opportuneness is obtained by equating this minimum temperature to the windfall temperature t_{w} and solving for the horizon of uncertainty, resulting in:

$$\widehat{\beta}_2 = \frac{k\left(\overline{T} + \dfrac{\widetilde{g}}{2k} - t_{\mathrm{w}}\right)}{\sqrt{\eta^T W^{-1}\eta}} \tag{621}$$

unless this expression is negative, in which case the opportuneness function takes the value zero.

Solution for problem 15. (p.11)

We first consider the **robustness.** The displacement at time t in response to input function $u(t)$ is:

$$x_u(t) \quad = \quad \frac{1}{m\omega}\int_0^t u(\tau)\sin\omega(t-\tau)\,\mathrm{d}\tau \tag{622}$$

$$= \underbrace{\frac{1}{m\omega}\int_0^t \widetilde{u}(\tau)\sin\omega(t-\tau)\,\mathrm{d}\tau}_{\widetilde{x}(t)} + \sum_{n=1}^{N}\phi_n \underbrace{\frac{1}{m\omega}\int_0^t \sin\frac{n\pi\tau}{T}\sin\omega(t-\tau)\,\mathrm{d}\tau}_{z_n} \tag{623}$$

$$= \widetilde{x}(t) + \phi^T z \tag{624}$$

The quadratic failure condition is equivalent to two other conditions:

$$x^2 \ge E_\mathrm{c} \quad\Longleftrightarrow\quad x \ge \sqrt{E_\mathrm{c}} \text{ or } x \le -\sqrt{E_\mathrm{c}} \tag{625}$$

Consequently, the robustness of design (m, k) with requirement E_c is:

$$\widehat{h}(m,k,E_\mathrm{c}) = \max\left\{h:\ \max_{u\in\mathcal{U}(h,\widetilde{u})} x_u \le \sqrt{E_\mathrm{c}} \ \text{ and } \ \min_{u\in\mathcal{U}(h,\widetilde{u})} x_u \ge -\sqrt{E_\mathrm{c}}\right\} \tag{626}$$

So we must find the extreme values of x_u up to horizon of uncertainty h. Consider the following sub-problem:

$$\max \phi^T z \quad \text{subject to} \quad \phi^T W \phi = h^2 \tag{627}$$

Define the objective function:

$$H = \phi^T z + \lambda(h^2 - \phi^T W \phi) \tag{628}$$

Conditions for extrema are:

$$0 = \frac{\partial H}{\partial \phi} = z - 2\lambda W\phi \quad\Longrightarrow\quad \phi = \frac{1}{2\lambda}W^{-1}z \tag{629}$$

The constraint implies:

$$h^2 = \frac{1}{4\lambda^2}z^T W^{-1} W W^{-1} z \quad\Longrightarrow\quad \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{z^T W^{-1} z}} \tag{630}$$

Hence the extremizing Fourier coefficients are:

$$\phi = \frac{\pm h}{\sqrt{z^T W^{-1} z}} W^{-1} z \tag{631}$$

Thus:

$$\max_{u\in\mathcal{U}(h,\widetilde{u})} \phi^T z = \pm h\sqrt{z^T W^{-1} z} \tag{632}$$

Hence:

$$\max_{u\in\mathcal{U}(h,\widetilde{u})} x_u = \widetilde{x} \pm h\sqrt{z^T W^{-1} z} \tag{633}$$

Hence, provided it is non-negative, the robustness is the greatest h satisfying:

$$\widetilde{x} + h\sqrt{z^T W^{-1} z} \le \sqrt{E_\mathrm{c}} \quad\Longrightarrow\quad \widehat{h}_+ = \frac{\sqrt{E_\mathrm{c}} - \widetilde{x}}{\sqrt{z^T W^{-1} z}} \tag{634}$$

and:

$$\widetilde{x} - h\sqrt{z^T W^{-1} z} \ge -\sqrt{E_\mathrm{c}} \quad\Longrightarrow\quad \widehat{h}_- = \frac{\sqrt{E_\mathrm{c}} + \widetilde{x}}{\sqrt{z^T W^{-1} z}} \tag{635}$$

Hence the robustness is:

$$\widehat{h}(m,k,E_\mathrm{c}) = \max\left[0,\ \min(\widehat{h}_+,\ \widehat{h}_-)\right] \tag{636}$$

In fig. 23 we show numerically calculated robustness curves, showing crossing of curves and reversal of design preference.

We now consider the **opportuneness.** The performance requirement upon which the robustness is based is:

$$-\sqrt{E_\mathrm{c}} \le x \le \sqrt{E_\mathrm{c}} \tag{637}$$

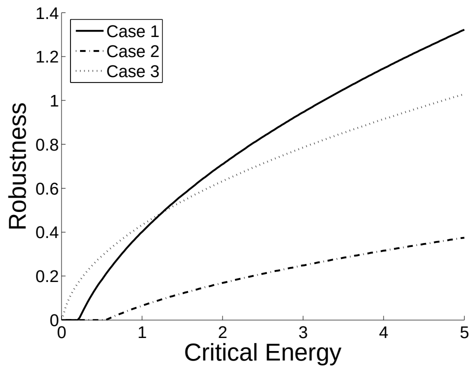

Figure 23: Robustness curves with problem 15. $\omega = 1,\ 1.2,\ 2$. $m = 1$, $\widetilde{u}(t) = \sin \omega_i t$, $\omega_i = 0.7$, $T = 5$.

That is, we require both of these inequalities to hold when we evaluate the robustness. For the opportuneness we require either (or both) of the following inequalities to be possible:

$$-\sqrt{E_{\rm w}} \le x \le \sqrt{E_{\rm w}} \tag{638}$$

where $E_{\rm w} < E_{\rm c}$. Thus, in analogy to eq.(626), the opportuneness function for design (m, k) with windfalling aspiration $E_{\rm w}$ is:

$$\widehat{\beta}(m, k, E_{\rm w}) = \min \left\{ h : \ \min_{u \in \mathcal{U}(h, \widetilde{u})} x_u \le \sqrt{E_{\rm w}} \ \ \text{or} \ \ \max_{u \in \mathcal{U}(h, \widetilde{u})} x_u \ge -\sqrt{E_{\rm w}} \right\} \tag{639}$$

Hence, provided it is non-negative, the opportuneness is the least h satisfying:

$$\widetilde{x} - h\sqrt{z^T W^{-1} z} \le \sqrt{E_{\rm w}} \quad \Longrightarrow \quad \widehat{\beta}_+ = \frac{\widetilde{x} - \sqrt{E_{\rm w}}}{\sqrt{z^T W^{-1} z}} \tag{640}$$

or:

$$\widetilde{x} + h\sqrt{z^T W^{-1} z} \ge -\sqrt{E_{\rm w}} \quad \Longrightarrow \quad \widehat{\beta}_- = \frac{-\sqrt{E_{\rm w}} - \widetilde{x}}{\sqrt{z^T W^{-1} z}} \tag{641}$$

Hence the opportuneness is:

$$\widehat{\beta}(m, k, E_{\rm c}) = \max \left[0,\ \min(\widehat{\beta}_+,\ \widehat{\beta}_-) \right] \tag{642}$$