Problem Set on
Info-Gap Uncertainty

List of problems.

1. **Formulate info-gap models of uncertainty.** (p.8) Formulate an info-gap model for uncertainty in the cost of raw materials in each of the following scenarios. Explain your choice and specify any additional information needed to verify your selection.

   (a) The cost of raw material varies erratically throughout the year.

   (b) The cost of raw material varies in a reasonably predictable manner throughout the year, except during the summer months when large erratic fluctuations are observed.

   (c) The cost of raw material varies in an unknown but gradual manner throughout the year.

   (d) The cost of raw material increases in an unknown but gradual manner throughout the year.

   (e) The cost of raw material varies in a reasonably predictable manner throughout the year, subject to occasional severe though short-lived excursions.

   (f) The cost of raw material varies randomly throughout the year, where the estimated pdf at time t is normal with mean and variance $\mu(t)$ and $\sigma^2(t)$. The tails of this pdf are highly uncertain.

2. **Formulate info-gap models of uncertainty, continued,** (p.8) Repeat the info-gap formulations in items 1a–1e where the phrase "cost of raw material" is replaced with "costs of N different raw materials". That is, we now consider an uncertain vector function rather than an uncertain scalar function.

3. **Some info-gap models.** (p.9) Consider the following info-gap models for uncertain scalar functions $u(t)$ defined on the domain $0 \le t < \infty$:

   Energy bound:
   $$\mathcal{U}_1(h, \widetilde{u}) = \left\{ u(t) : \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \tag{1}$$

   Uniform bound:
   $$\mathcal{U}_2(h, \widetilde{u}) = \{ u(t) : |u(t) - \widetilde{u}(t)| \le h \}, \quad h \ge 0 \tag{2}$$

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(a) At any positive value of the uncertainty parameter h neither of the sets, neither $\mathcal{U}_1(h,\widetilde{u})$ nor $\mathcal{U}_2(h,\widetilde{u})$, is contained in the other. For each set, find an element which *it contains* and which is *not contained* in the other set. With these results, explain the different uses or interpretations of these two info-gap models of uncertainty.

(b) Despite the fact that neither of the info-gap models is included in the other, there are nonetheless an infinity of functions which belong to both. Consider the function:

$$u(t) = \widetilde{u}(t) + \mathrm{e}^{-\lambda t} \tag{3}$$

where λ is a positive constant much less than unity. Show that:

$$u(t) \in \mathcal{U}_1(h,\widetilde{u}) \quad \text{for all} \quad h \geq \frac{1}{\sqrt{2\lambda}} \tag{4}$$

$$u(t) \in \mathcal{U}_2(h,\widetilde{u}) \quad \text{for all} \quad h \geq 1 \tag{5}$$

Note that $\frac{1}{\sqrt{2\lambda}} \gg 1$. What does this imply about the coherence or compatibility of this function ($u(t)$ in eq.(3)) with each of the info-gap models?

4. **Fourier expansion.** (p.10) Consider the function:

$$f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \\ -1, & -1 \leq x < 0 \end{cases} \tag{6}$$

(a) Express $f(x)$ as a Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin n\pi x \tag{7}$$

That is, find the Fourier coefficients c_n. (Hint: exploit the orthogonality of the sine functions.)

(b) Draw the approximation to $f(x)$:

$$f_K(x) = \sum_{n=1}^{K} c_n \sin n\pi x \tag{8}$$

for $K = 5$, 10, 20, 100.

(c) Can $f(x)$ be represented as a Fourier cosine series? Explain.

5. **Properties of ellipsoids,** (p.10).

(a) Draw the ellipsoid:

$$ax^2 + by^2 = 1 \tag{9}$$

What are the directions and lengths of the principal axes?

(b) Draw the ellipsoid:

$$2x^2 + 2xy + 2y^2 = 1 \tag{10}$$

What are the directions and lengths of the principal axes?

(c) Given an N-dimensional ellipsoid:

$$x^T W x = 1 \quad (11)$$

where W is a real, symmetric, positive definite matrix. What are the directions and lengths of the principal axes?

(Hint: start with part (c).)

6. ‡[1] **More info-gap models.** (p.) Consider the following info-gap models for uncertain scalar functions $u(t)$ defined on the domain $0 \le t \le T$:

Uniform bound:

$$\mathcal{U}_1(h, \widetilde{u}) = \{u(t) : \ |u(t) - \widetilde{u}(t)| \le h\}, \quad h \ge 0 \quad (12)$$

Fourier ellipsoid bound:

$$u(t) = \widetilde{u}(t) + \sum_{n=1}^{N} \left[a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right] \quad (13)$$

$$= \widetilde{u}(t) + c^T \phi(t) \quad (14)$$

where c is the vector of uncertain Fourier coefficients and $\phi(t)$ is the corresponding vector of cosine and sine functions. A Fourier ellipsoid bound info-gap model is:

$$\mathcal{U}_2(h, \widetilde{u}) = \left\{ u(t) = \widetilde{u}(t) + c^T \phi(t) : \ c^T c \le h^2 \right\}, \quad h \ge 0 \quad (15)$$

(a) At any fixed positive value of the uncertainty parameter h, is one of the sets, $\mathcal{U}_1(h, \widetilde{u})$ or $\mathcal{U}_2(h, \widetilde{u})$, contained in the other?

(b) Show that, at all values of the uncertainty parameter h, the following inclusion holds:

$$\mathcal{U}_2(h, \widetilde{u}) \ \subset \ \mathcal{U}_1(h\sqrt{N}, \widetilde{u}) \quad (16)$$

where N is the number of modes in the Fourier expansion in eq.(13). What is the interpretation of this inclusion? What is the significance of this constant, $\sqrt{N}$, which scales the uncertainty parameter? That is, why does the scale parameter depend on the order of the Fourier expansion?

7. ‡ **Convex info-gap models.** (p.) Show that the following are *convex* info-gap models.

Energy bound:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \ \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \quad (17)$$

Uniform bound:

$$\mathcal{U}(h, \widetilde{u}) = \{u(t) : \ |u(t) - \widetilde{u}(t)| \le h\}, \quad h \ge 0 \quad (18)$$

Slope bound:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \ u(0) = 0, \ \frac{\mathrm{d}u(t)}{\mathrm{d}t} \le h \right\}, \quad h \ge 0 \quad (19)$$

Ellipsoid bound:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u : \ (u - \widetilde{u})^T W (u - \widetilde{u}) \le h^2 \right\}, \quad h \ge 0 \quad (20)$$

[1] Problems marked with a double-dagger, ‡, contain more advanced material.

where W is a real, symmetric, positive definite matrix.

Interval bound:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u : \left| \frac{u_n - \widetilde{u}_n}{\widetilde{u}_n} \right| \le h, \ n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{21}$$

Hint: A set S is convex if:

$$x \in S \text{ and } y \in S \quad \text{implies} \quad \gamma x + (1-\gamma) y \in S \tag{22}$$

for all $\gamma \in (0, 1)$.

8. **Two-counter riddle,** (p.14) Following is a riddle posed and "solved" by Lewis Carroll[2] Is his argument correct? If not, where are his errors?

> "A bag contains 2 counters, as to which nothing is known except that each is either black or white. Ascertain their colours without taking them out of the bag."

Answer: "One is black, and the other white."

Solution:

> "We know that, if a bag contained 3 counters, 2 being black and one white, the chance of drawing a black one would be 2/3— and that any *other* state of things would *not* give this chance.
>
> "Now the chances, that the given bag contains (h) BB, (β) BW, (γ) WW, are respectively 1/4, 1/2, 1/4.
>
> "Add a black counter.
>
> "Then the chances, that it contains (h) BBB, (β) BWB, (γ) WWB, are, as before, 1/4, 1/2, 1/4.
>
> "Hence the chance, of now drawing a black one,
>
> $$= \frac{1}{4} \cdot 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1}{3} = \frac{2}{3}$$
>
> "Hence the bag now contains BBW (since any *other* state of things would *not* give this chance.
>
> "Hence, before the black counter was added, it contained BW, i.e. one black counter and one white."

9. **Modification of the 3-box problem.** (p.15) Suppose that we have imprecise probabilistic information about the location of the prize. Let $\overline{p}_i$ be the best guess of the probability that the prize is in the ith box, where $\overline{p}_1 + \overline{p}_2 + \overline{p}_3 = 1$ and $\overline{p}_1 \ge \overline{p}_2 \ge \overline{p}_3$. However, these values are uncertain, and are constrained to the info-gap model:

$$\mathcal{P}(h) = \left\{ (p_1, p_2) : \ p_i \ge 0, \ p_1 + p_2 \le 1, \ (p_1 - \overline{p}_1)^2 + (p_2 - \overline{p}_2)^2 \le h^2 \right\}, \quad h \ge 0 \tag{23}$$

$\mathcal{P}(h)$ is an unbounded family of sets of prior distributions. Each set contains the nominal distribution $\overline{p}$. The sets become more inclusive as h increases.

Given values $(\overline{p}_1, \overline{p}_2, \overline{p}_3)$, we will choose the box 1 whose estimated probability is highest.

[2] L. Carroll 1895, *Pillow Problems,* Re-issued by Dover Press, New York, 1958, riddle 72.

(a) Evaluate the robustness of this decision with respect to the uncertainty h. That is, what is the greatest value of h (call this maximum $\widehat{h}$) such that the decision is the same for any doublet $(p_1, p_2) \in \mathcal{P}(h)$, if $h \le \widehat{h}$? In other words, how much uncertainty can the decision algorithm tolerate without altering the decision?

(b) Show that the robustness is zero when $\overline{p}_1 = \overline{p}_2 = \overline{p}_3 = 1/3$. Discuss the implications of this for the original 3-box problem.

10. **Choose the larger number,** (p.16) Two different real numbers, x_1 and x_2, are chosen by an algorithm unknown to you. One of these numbers, call it x_r, is revealed to you, where you know[3] that the probability that $x_\mathrm{r} = x_1$ is 0.5. You must decide if x_r is the smaller or the larger of the two numbers.

    For example, two systems have an attribute (e.g. lifetime, reliability, etc.) with values x_1 and x_2, but we are able to test and estimate the attribute of only one system. We must decide if the revealed attribute is the smaller or the larger of the two, where we have chosen the system to test by a throw of a fair coin.

    (a) Let $q(y)$ be a pdf which is positive on all real numbers. Consider the following decision rule:[4]

    (a) Draw a random number, y, distributed according to $q(y)$.
    (b) If $y \ge x_\mathrm{r}$ then decide that x_r is the smaller of the two x_i.
    (c) If $y < x_\mathrm{r}$ then decide that x_r is the larger of the two x_i.

    Show that the probability of successful decision with this rule is strictly greater than 1/2. What is an intuitive explanation of why this algorithm works?

    (b) Suppose that we have a rough guess of the pdf by which the x_i are chosen. Specifically, suppose we think they are drawn from a joint pdf which is something like $\widetilde{p}(x_1, x_2)$. How should we represent the uncertainty in the pdf? How should we choose the distribution $q(y)$ which will be used to decide according to the algorithm in part (a)? Formulate the robustness for the probability of successful decision with $q(y)$.

    (c) We now use the result of part (b) in a very simple special case. Suppose we know that x_1 and x_2 are chosen independently from an exponential distribution, $p(x) = \lambda \mathrm{e}^{-\lambda x}$, $x \ge 0$. Suppose our best guess of the coefficient is $\widetilde{\lambda}$ but this guess is very uncertain. Now use a fractional-error info-gap model for uncertainty in the exponential coefficient of the pdf by which the x_i are chosen:

$$\mathcal{U}(h, \widetilde{p}) = \left\{ p(x) = \lambda \mathrm{e}^{-\lambda x} :\ \max[0,\ (1-h)\widetilde{\lambda}] \le \lambda \le (1+h)\widetilde{\lambda} \right\}, \quad h \ge 0 \tag{24}$$

    Furthermore, assume that the pdf used for deciding is also exponential: $q(y) = \gamma \mathrm{e}^{-\gamma y}$. Derive the robustness function (or its inverse) and explore the choice of γ.

    (d) Demonstrate the decision algorithm described in part (a) by simulation. Draw N pairs of numbers, (x_1, x_2), independently from a 'generating' pdf $p(x)$ of your choice. Use the decision algorithm described above with a 'deciding' pdf $q(y)$ of your choice. The theoretical probability of success for a pair (x_1, x_2) is $P_\mathrm{s}(x_1, x_2)$ (this was derived in part (a)). The *average theoretical probability of success,* over the N draws, is:

$$\overline{P}_\mathrm{s} = \frac{1}{N} \sum_{i=1}^{N} P_\mathrm{s}(x_1, x_2) \tag{25}$$

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[3] It is very important that we know the probability is equal. Otherwise, invoking the principle of indifference would lead to a contradiction, as in the typical 2-envelope problem.

[4] The algorithm was proposed by Thomas M. Cover, 1987, Pick the largest number, chapter 5.1 in T. Cover and B. Gopinath, 1987, *Open Problems in Communication and Computation,* Springer-Verlag, Berlin. See also:
∘ Snapp, Robert R., 2005, Tom Cover's Number Guessing Game, http:/www.cems.uvm.edu/˜snapp/teaching/ cover-sproblem.pdf.
∘ http:/blog.xkcd.com/2010/02/09/math-puzzle.
∘ Yakov Ben-Haim, 2011, Two for the Price of One: Info-Gap Robustness of the 1-Test Algorithm, 7th Intl Symp on Imprecise Probabilities and their Applications, Innsbruck, Austria. Available at: http:/info-gap.com/content.php?id=22

The *empirical probability of success,* F, is the fraction of the N draws in which the q-distribution decision algorithm decides correctly: deciding 'smaller' when $x_{\rm r}$ is the smaller between x_1 and x_2 and deciding 'larger' when $x_{\rm r}$ is the larger between x_1 and x_2. Show that the empirical and the average theoretical probabilities of success agree.

(e) Consider n systems with values $x_1, \ldots, x_n$, which are all different. Suppose that $m < n$ of these values are revealed, with equal probabilities for each system to be revealed. Can the algorithm of part (a) be generalized? Formulate and study the robustness of the choice of the decision pdf.