Lecture Notes on Hybrid Uncertainties

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Source material:
• Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty*, 2nd edition, Academic Press, chapter 10.
• Yakov Ben-Haim, 1996, *Robust Reliability in the Mechanical Sciences,* Springer, chap. 8.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

Contents

¶ Sometimes one has both **probabilistic** and **info-gap** information about the uncertainties.

¶ Neither is sufficient to fully characterize the uncertainty.

¶ We will consider three situations:
- Info-gap uncertainty and the Poisson process.
- Uncertain probability distributions embedded in an info-gap model.
- Probabilistic info-gap horizon of uncertainty.

1 Info-Gap Uncertainty in a Poisson Process

1.1 Poisson and Info-Gap Uncertainties

¶ Many complex events such as earthquakes, currency crashes, or other extreme disturbances have **two distinct time constants:**

1. The events recur infrequently over time.

   That is, on the **long time scale,** θ, they can be thought of as distinct points.

2. The temporal variation during an event is both important and unknown.

   That is, on the **short time scale,** t, they are complex and unknown.

¶ A common and often reliable statistical datum on the long time scale is:
Average rate of recurrence of a rare event over a long duration θ.

¶ The **poisson process** is a good probabilistic model for long durations if:

1. The occurrence of distinct events is statistically independent.

2. The average number of events per unit of time is constant.

¶ With these two assumptions, the probability of exactly n events in a duration θ is given by the Poisson distribution:

$$P_n(\theta) = \frac{(\lambda\theta)^n e^{-\lambda\theta}}{n!}, \quad n = 0, 1, 2, \ldots \tag{1}$$

¶ This is valid for representing distributions in space as well as in time.

¶ The mean number of events in duration θ is:

$$\mathrm{E}[n(\theta)] = \lambda\theta \tag{2}$$

¶ Thus $\lambda =$ mean number of events per unit time.

¶ An info-gap model is a good representation of the uncertain variation of the temporal waveform during an event.

1.2 Shock Loading of a Dynamical System

¶ Dynamical system:
- ○ t = short time scale.
- ○ $x_u(t)$ = state vector.
- ○ $u(t)$ = Severe transient load vector.

¶ Damage due to loads:
- ○ Severe loads recur infrequently, causing damage.
- ○ Damage depends on the short-time-scale dynamics.
- ○ Damage accumulates from each event, until the system fails.

¶ System model:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bu(t), \quad x(0) = 0 \tag{3}$$

A and B are known constant matrices.

¶ Cumulative energy-bound load-uncertainty model:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \int_0^\infty [u(t) - \widetilde{u}(t)]^T W [u(t) - \widetilde{u}(t)] \, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \tag{4}$$

W is a known, real, symmetric, positive definite matrix.

¶ Small increment of damage resulting from one event:

$$\delta_u = \gamma \left[\psi^T x_u(t) \right]^\mu \tag{5}$$

γ and μ are known, positive constants.
ψ is a known projection vector.

¶ Poisson probability, $P_n(\theta)$, of n transient events in a long interval of duration θ, eq.(1). Single known parameter, λ.

¶ Failure occurs if the **cumulative damage** exceeds Δ_c.

1.3 Robustness Function: I

¶ Failure occurs in n events if the
cumulative damage exceeds the critical value Δ_c.

¶ The robustness to $n > 0$ events, $\widehat{h}_n$,
is the greatest value of the uncertainty parameter h
such that failure cannot occur in n events:

$$\widehat{h}_n = \max \left\{ h : \; n \max_{u \in \mathcal{U}(h, \widetilde{u})} \delta_u(t) \leq \Delta_c \right\} \tag{6}$$

We note that $\widehat{h}_n$ is meaningful for $n > 0$. Failure cannot occur if damage-inducing events do not occur.

1.4 Maximal Increment of Damage

¶ In order to evaluate the robustness function we must find the maximum increment of damage in a single event, up to uncertainty h.

¶ This requires the maximum projected response.

¶ The response to input $u(t)$ is:

$$x_u(t) = \int_0^t \mathrm{e}^{A(t-\tau)} B u(\tau)\, \mathrm{d}\tau \tag{7}$$

¶ The deviation of the projected response is:

$$\psi^T \left[x_u(t) - x_{\widetilde{u}}(t)\right] = \int_0^t \psi^T \mathrm{e}^{A(t-\tau)} B \left[u(\tau) - \widetilde{u}(\tau)\right] \, \mathrm{d}\tau \tag{8}$$

$$= \int_0^t \psi^T \mathrm{e}^{A(t-\tau)} B W^{-1/2} W^{1/2} \left[u(\tau) - \widetilde{u}(\tau)\right] \, \mathrm{d}\tau \tag{9}$$

$$= \int_0^t \zeta^T (t-\tau) W^{1/2} \left[u(\tau) - \widetilde{u}(\tau)\right] \, \mathrm{d}\tau \tag{10}$$

where we have defined the vector:

$$\zeta^T(t) = \psi^T \mathrm{e}^{At} B W^{-1/2} \tag{11}$$

¶ The maximum projected response up to uncertainty h is:

$$\max_{u \in \mathcal{U}(h,\widetilde{u})} \psi^T \left[x_u(t) - x_{\widetilde{u}}(t)\right] = h \underbrace{\sqrt{\int_0^t \zeta^T(\tau) \zeta(\tau)\, \mathrm{d}\tau}}_{Z(t)} \tag{12}$$

which defines the known function $Z(t)$.
(Hint: use the Cauchy inequality, and then the Schwarz inequality.)
¶ Now, combining eqs.(5) and (12), the maximum increment of damage in a single transient event, up to uncertainty h, is:

$$\max_{u \in \mathcal{U}(h,\widetilde{u})} \delta_u(t) = \gamma \left[\psi^T x_{\widetilde{u}}(t) + h Z(t)\right]^{\mu} \tag{13}$$

1.5 Robustness Function: II

¶ Failure occurs in n events if the
cumulative damage exceeds the critical value Δ_c.

¶ As explained in section 1.3, the robustness to $n > 0$ events, $\widehat{h}_n$,
is the greatest value of the uncertainty parameter h
such that failure cannot occur in n events:

$$\widehat{h}_n = \max \left\{ h : \; n \max_{u \in \mathcal{U}(h, \widetilde{u})} \delta_u(t) \leq \Delta_c \right\} \tag{14}$$

We note that $\widehat{h}_n$ is meaningful for $n > 0$. Failure cannot occur if damage-inducing events do not occur.

¶ Equate max cumulative damage to Δ_c:

$$n \max_{u \in \mathcal{U}(h, \widetilde{u})} \delta_u(t) = \Delta_c \tag{15}$$

Now solve for h to find the robustness to n transients:

$$\widehat{h}_n = \frac{(\Delta_c / n\gamma)^{1/\mu} - \psi^T x_{\widetilde{u}}(t)}{Z(t)}, \quad n = 1, 2, \ldots \tag{16}$$

or $\widehat{h}_n = 0$ if this is negative.

¶ n is a Poisson random variable.
Therefore $\widehat{h}_n$ is also a Poisson random variable.

¶ Randomization: concise combination of
info-gap and probabilistic information.

$$\widehat{h}(\theta) \quad = \quad \frac{1}{1-P_0(\theta)} \sum_{n=1}^{\infty} \widehat{h}_n P_n(\theta) \tag{17}$$

We are usually interested in long durations θ for which:

$$P_0(\theta) = \mathrm{e}^{-\lambda\theta} \ll 1 \tag{18}$$

¶ $\widehat{h}(\theta)$ is a decision function, since "bigger is better".

¶ Let q be the vector of decision variables. We will write $\widehat{h}(q, \Delta_{\rm c})$.

¶ The optimal optimal decision vector $\widehat{q}_{\rm c}(\Delta_{\rm c})$:

$$\widehat{q}_{\rm c}(\Delta_{\rm c}) = \arg\max_{q \in \mathcal{Q}} \widehat{h}(q, \Delta_{\rm c}) \tag{19}$$

$\mathcal{Q}$ = set of available decisions.

¶ Both robustness functions:
$\widehat{h}(q, \Delta_{\rm c})$ and $\widehat{h}(\widehat{q}_{\rm c}(\Delta_{\rm c}), \Delta_{\rm c})$,
display the usual trade-off of immunity versus reward.

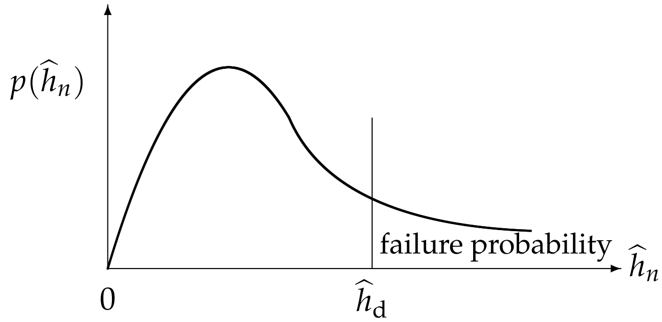

Figure 1: Illustration of failure probability for eq.(20).

¶ Different approach: Optimize probability distribution of $\widehat{h}_n$.

- ◦ Let $\widehat{h}_{\rm d}$ be a desired or demanded value of robustness.
- ◦ Choose q to maximize the probability of those $\widehat{h}_n(q)$'s which exceed the demanded value $\widehat{h}_{\rm d}$:

$$\widehat{q}(\widehat{h}_{\rm d}) = \arg\max_{q \in \mathcal{Q}} \sum_{\widehat{h}_n(q) \ge \widehat{h}_{\rm d}} P_n(\theta) \tag{20}$$

Let us examine the condition:

$$\widehat{h}_n(q) \ge \widehat{h}_{\rm d} \tag{21}$$

From eq.(16) this becomes:

$$\left(\frac{\Delta_{\rm c}}{n\gamma}\right)^{1/\mu} \ge \psi^T x_{\widetilde{u}}(t) + \widehat{h}_{\rm d} Z(t) \tag{22}$$

Solving for n:

$$n \le \frac{\Delta_{\rm c}}{\gamma \left[\psi^T x_{\widetilde{u}}(t) + \widehat{h}_{\rm d} Z(t)\right]^{\mu}} \tag{23}$$

We maximize the probability that condition (21) holds if we choose q to minimize $\psi^T x_{\widetilde{u}}(t) + \widehat{h}_{\rm d} Z(t)$.

2 Embedded Probability Densities

¶ We consider the following situation:
- ◦ u is uncertain.
- ◦ The uncertainty in u is represented by a pdf $p(u)$.
- ◦ $p(u)$ is uncertain.
- ◦ The uncertainty in $p(u)$ is represented by an info-gap model.

2.1 Formulation: Dynamical System

¶ Variables:
- ◦ u = uncertain input to a system.
- ◦ x_u = response to input u.
- ◦ $p(u)$ = pdf of u; imperfectly known.
- ◦ $\widetilde{p}(u) =$ nominal pdf of u; known.
- ◦ $\mathcal{U}(h, \widetilde{p}),\ h \geq 0$: info-gap model for uncertainty of p.

¶ Failure occurs if:

$$f(x_u) > x_c \tag{24}$$

¶ For any pdf $p(u)$, the probability of failure is:

$$P_f(p) = \text{Prob}\,(f(x_u) > x_c \,|\, p) \tag{25}$$

$$= \int_{f(x_u) > x_c} p(u)\,\mathrm{d}u \tag{26}$$

¶ We want:

$$P_f(p) \leq P_c \tag{27}$$

¶ We cannot reliably calculate $P_f(p)$ because p is uncertain.
¶ We **can** calculate the robustness, to uncertainty in $p(u)$, of the failure probability:

$$\widehat{h}(P_c) = \max \left\{ h : \max_{p \in \mathcal{U}(h, \widetilde{p})} P_f(p) \leq P_c \right\} \tag{28}$$

This is an ordinary robustness function for uncertainty in p.
If $\widehat{h}(P_c)$ is large then we have confidence, despite the info-gaps in the pdf, that the failure probability will not exceed P_c.

2.2 Example: 1-D Dynamic System

¶ 1-D system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax(t) + Bu(t), \quad x(0) = 0 \tag{29}$$

A and B are known constant scalars.

¶ Variables:
- ○ u = input.
  = constant random variable in $[0, T]$. Zero elsewhere.
- ○ $p(u)$ = pdf of u.
- ○ $\widetilde{p}(u)$ = best-estimate of the probability density of u.
  = $\mathcal{N}(0, \sigma^2)$.

¶ Uncertainty in $p(u)$:
- ○ Evidence for $\widetilde{p}$ is quite good up to about k standard deviations.
- ○ Beyond $k\sigma$ the fractional deviation of p from $\widetilde{p}$ varies.
- ○ An info-gap model for uncertainty in p is:

$$\mathcal{U}(h, \widetilde{p}) = \left\{ p(u) : \begin{array}{l} p(u) \geq 0, \int p(u)\,\mathrm{d}u = 1, \\ |p(u) - \widetilde{p}(u)| \leq h\widetilde{p}(u) \text{ if } |u| \geq k\sigma \\ p(u) = c\widetilde{p}(u) \text{ if } |u| < k\sigma \end{array} \right\}, \quad h \geq 0 \tag{30}$$

c is a normalization constant for each density $p(u)$.

¶ System response at end of nominal load:

$$x_u(T) = \frac{uB\left(\mathrm{e}^{AT} - 1\right)}{A} \tag{31}$$

¶ Failure criterion:

$$|x_u(T)| > x_\mathrm{c} \tag{32}$$

¶ Probability of failure, given density $p(u)$, is:

$$P_\mathrm{f}(p) = \mathrm{Prob}\left(|x_u(T)| > x_\mathrm{c} \mid p\right) \tag{33}$$

$$= \mathrm{Prob}\left(|u| > \eta x_\mathrm{c}\right) \tag{34}$$

where we have defined:

$$\eta = \frac{A}{B\left(\mathrm{e}^{AT} - 1\right)} \tag{35}$$

¶ As before, we desire:

$$P_\mathrm{f}(p) \leq P_\mathrm{c} \tag{36}$$

¶ Simplifying assumption:

$$\eta x_{\mathrm{c}} \geq k\sigma \tag{37}$$

¶ To evaluate the robustness function we must find maximum failure probability.

¶ The maximum on the upper tail is:

$$\max_{p\in\mathcal{U}(h,\widetilde{p})} \int_{\eta x_{\mathrm{c}}}^{\infty} p(u)\,\mathrm{d}u = \int_{\eta x_{\mathrm{c}}}^{\infty} \widetilde{p}(u)(1+h)\,\mathrm{d}u \tag{38}$$

$$= (1+h)\left[1-\Phi\left(\frac{\eta x_{\mathrm{c}}}{\sigma}\right)\right] \tag{39}$$

$\Phi(\cdot)$ is the standard normal probability distribution function.

¶ The maximum on the lower tail is the same, so:

$$\max_{p\in\mathcal{U}(h,\widetilde{p})} P_{\mathrm{f}}(p) = 2(1+h)\left[1-\Phi\left(\frac{\eta x_{\mathrm{c}}}{\sigma}\right)\right] \tag{40}$$

¶ We have assumed that h is small enough so that this is no greater than one. This is assured, for some non-negative h, if the nominal density, $\widetilde{p}(u)$, entails acceptable probability of failure, which requires that:

$$2\left[1-\Phi\left(\frac{\eta x_{\mathrm{c}}}{\sigma}\right)\right] \leq P_{\mathrm{c}} \tag{41}$$

¶ To find $\widehat{h}$ from eq.(28) on p.10, equate eq.(40) to P_{c}, and solve for h:

$$\widehat{h}(P_{\mathrm{c}}) = \frac{P_{\mathrm{c}}}{2\left[1-\Phi\left(\frac{\eta x_{\mathrm{c}}}{\sigma}\right)\right]} - 1 \tag{42}$$

2.3 Example: Static Poisson Queuing I

¶ **Queuing and timing problems:**

• Match server rate to client-arrival rate.

◦ Inventory problems: keep stock available and fresh.

◦ Digital communications synchronization.

• Tracking random events.

¶ **The System:**

• Server able to handle r clients per day.

• Clients accumulate during the night; no new clients arrive during working hours.

• $n =$ number of clients waiting in morning.

• Clients arrive randomly and independently with constant mean rate, so n is a Poisson random variable:

$$P_n(\lambda) = \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!}, \quad n = 0, 1, \ldots \tag{43}$$

¶ **Uncertainty:**

• $\lambda =$ average number of clients per day. Non-negative

• $\widetilde{\lambda}$ = best estimate of λ.

• λ erratically variable, and represented by fractional-error info-gap model:

Approximately:

$$\left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \leq h, \quad h \geq 0 \tag{44}$$

More precisely:

$$\mathcal{U}(h, \widetilde{\lambda}) = \left\{ \lambda : \ \max[0, \ (1-h)\widetilde{\lambda}] \leq \lambda \leq (1+h)\widetilde{\lambda} \right\}, \quad h \geq 0 \tag{45}$$

¶ **The Question:**

• Manager does not want:

◦ Clients who are not handled on the day of arrival: r too small.

◦ Unused client-handling capability: r too large.

• What value of r should be adopted?

¶ **Loss function:**

• Probability of Not Serving s_2 or more clients is:

$$\pi_{\mathrm{ns}}(r,\lambda) = \sum_{n=r+s_2}^{\infty} P_n(\lambda) \tag{46}$$

• Probability of Unused Capacity for handling s_1 or more clients is:

$$\pi_{\mathrm{uc}}(r,\lambda) = \sum_{n=0}^{r-s_1} P_n(\lambda) \tag{47}$$

• The loss function is:

$$\begin{aligned}
\pi_\ell(r,\lambda) &= \pi_{\mathrm{uc}}(r,\lambda) + \pi_{\mathrm{ns}}(r,\lambda) && (48)\\
&= \sum_{n=0}^{r-s_1} P_n(\lambda) + \sum_{n=r+s_2}^{\infty} P_n(\lambda) && (49)\\
&= 1 - \sum_{n=r-s_1+1}^{r+s_2-1} P_n(\lambda) && (50)\\
&= 1 - \mathrm{e}^{-\lambda} \sum_{n=r-s_1+1}^{r+s_2-1} \frac{\lambda^n}{n!} && (51)
\end{aligned}$$

• For instance, if $s_1 = s_2 = 1$:

$$\pi_\ell(r,\lambda) = 1 - P_r(\lambda) = 1 - \frac{\mathrm{e}^{-\lambda}\lambda^r}{r!} \tag{52}$$

¶ **Performance requirement:**

$$\pi_\ell(r,\lambda) \le \varepsilon \tag{53}$$

¶ **Robustness** of handling-capacity r to uncertainty in arrival rate λ:

$$\widehat{h}(r,\varepsilon) = \max\left\{ h :\ \left(\max_{\lambda \in \mathcal{U}(h,\widetilde{\lambda})} \pi_\ell(r,\lambda) \right) \le \varepsilon \right\} \tag{54}$$

¶ **Inner maximum** in eq.(54):

$$M(h) = \max_{\lambda \in \mathcal{U}(h,\widetilde{\lambda})} \pi_\ell(r,\lambda) \tag{55}$$

• $M(h)$ increases as h increases because $\mathcal{U}(h,\widetilde{\lambda})$ are nested sets:

$$\frac{\mathrm{d}M(h)}{\mathrm{d}h} \geq 0 \tag{56}$$

• $\widehat{h}(r,\varepsilon)$ is greatest h at which:

$$M(h) \leq \varepsilon \tag{57}$$

• Thus $\widehat{h}(r,\varepsilon)$ is greatest solution for h of (see fig. 2):

$$M(h) = \varepsilon \tag{58}$$

• In other words, $M(h)$ is the inverse of $\widehat{h}(r,\varepsilon)$:

$$M(h) = \varepsilon \quad \text{if and only if} \quad \widehat{h}(r,\varepsilon) = h \tag{59}$$

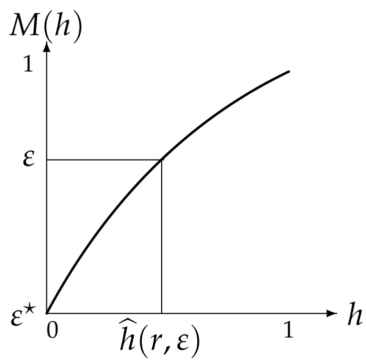

Figure 2: Illustration of the calculation of robustness.

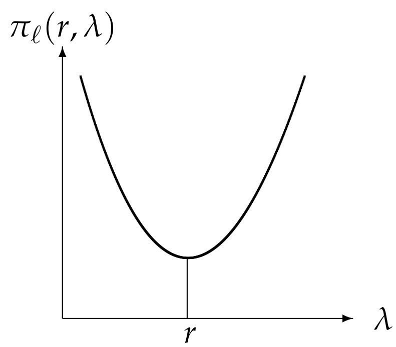

Figure 3: Schematic illustration of $\pi_\ell(r,\lambda)$.

¶ **Evaluating** $M(h)$:

• Consider $s_1 = s_2 = 1$, so $\pi_\ell(r,\lambda)$ in eq.(52), p.14, is:

$$\pi_\ell(r,\lambda) = 1 - \frac{\mathrm{e}^{-\lambda}\lambda^r}{r!} \tag{60}$$

• Note, as illustrated schematically in fig. 3, that:

$$\frac{\partial \pi_\ell}{\partial \lambda} = \frac{\mathrm{e}^{-\lambda}\lambda^{r-1}}{r!}(\lambda - r) \tag{61}$$

• Hence, $M(h)$ is obtained from eq.(60) with one or the other of the extreme λ values at horizon of uncertainty h. Denote these extreme values:

$$\lambda_+ = (1+h)\widetilde{\lambda} \tag{62}$$

$$\lambda_- = \max[0,\ (1-h)\widetilde{\lambda}] \tag{63}$$

• Hence:

$$M(h) = \max\left[\pi_\ell(r,\lambda_-),\ \pi_\ell(r,\lambda_+)\right] \tag{64}$$

¶ **Nominal loss function** for $s_1 = s_2 = 1$, eq.(60), p.15:

$$\varepsilon^\star = \pi_\ell(r, \widetilde{\lambda}) = 1 - \frac{e^{-\widetilde{\lambda}}\widetilde{\lambda}^r}{r!} \tag{65}$$

This estimate of the loss function is based on the best estimate of the client-arrival rate, $\widetilde{\lambda}$.

• Note that:

$$M(0) = \varepsilon^\star \tag{66}$$

• Thus, as in fig. 2, p.15:

$$\widehat{h}(r, \varepsilon^\star) = 0 \tag{67}$$

◦ The best estimate of the loss function has zero robustness.
◦ Only worse (larger) loss has positive robustness, as in fig. 2:

$$\varepsilon > \varepsilon' \quad \Longrightarrow \quad \widehat{h}(r, \varepsilon) \geq \widehat{h}(r, \varepsilon') \tag{68}$$

¶ **Optimizing the nominal loss function.**

• Optimal server size:

$$r^\star = \arg\min_r \pi_\ell(r, \widetilde{\lambda}) \tag{69}$$

• Anticipated loss function:

$$\varepsilon^{\rm opt} = \pi_\ell(r^\star, \widetilde{\lambda}) \tag{70}$$

• Robustness vanishes as in eq.(67):

$$\widehat{h}(r^\star, \varepsilon^{\rm opt}) = 0 \tag{71}$$

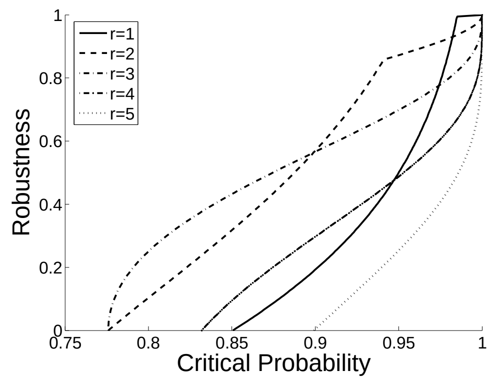

Figure 4: Robustness curves for $\widetilde{\lambda} = 3$ and $r = 1, 2, \ldots, 5$. $s_1 = s_2 = 1$.

¶ **Numerical example,** fig. 4.

• The best (but highly unreliable) estimate of the number of clients is $\widetilde{\lambda} = 3$.

• Fig. 4 shows robustness curves for server-capacities $r = 1, 2, \ldots, 5$.

• Recall the loss function, $\pi_\ell(r, \lambda)$, which is the probability of un-served clients or unused server capacity.

• Consider the loss function at the estimated number of clients, $\pi_\ell(r, \widetilde{\lambda})$, which is the x-intersect in fig. 4, shown in table 1:

r Server capacity	$M(0) = \pi_\ell(r, \widetilde{\lambda})$ Nominal loss function
1	0.85
2	0.78
3	0.78
4	0.83
5	0.90

Table 1: Nominal loss function for different server capacities.

• We want $\pi_\ell(r, \widetilde{\lambda})$ small, so, based on the best-estimate of the client-arrival rate, $\widetilde{\lambda}$, our preferences on values of r are:

$$3 \sim_{\text{n}} 2 \succ_{\text{n}} 4 \succ_{\text{n}} 1 \succ_{\text{n}} 5 \tag{72}$$

The subscript 'n' indicates that these are 'nominal' preferences.

• Now consider the preferences based on the robustness curves, $\succ_{\rm r}$.
  ∘ An r-value whose curve is further to the right has greater robustness.
  ∘ The following *strict dominances* are observed:

$$3 \succ_{\rm r} 4 \succ_{\rm r} 5 \tag{73}$$

$$2 \succ_{\rm r} 1 \succ_{\rm r} 5 \tag{74}$$

  ∘ The robust-satisficing preferences in eqs.(73) and (74) are consistent with, but weaker than, the nominal preferences in eq.(72).
• In fig. 4 we see 3 **crossing robustness curves.**
• Crossing of robustness curves implies preference reversal.
• Comparing nominal and robust-satisficing preferences, the differences are shown in table 2:

$\succ_{\rm n}$ Nominal preference	$\succ_{\rm r}$ robust-satisficing preference
$3 \sim_{\rm n} 2$	3 crosses 2
$3 \succ_{\rm n} 1$	3 crosses 1
$4 \succ_{\rm n} 1$	4 crosses 1

Table 2: Nominal loss function for different server capacities.

• For instance, compare $r = 2$ and $r = 3$ in fig. 4.
  ∘ For $\varepsilon < 0.9$: $\widehat{h}(3,\varepsilon) > \widehat{h}(2,\varepsilon) \implies 3 \succ_{\rm r} 2$.
  ∘ For $\varepsilon > 0.9$: $\widehat{h}(2,\varepsilon) > \widehat{h}(3,\varepsilon) \implies 2 \succ_{\rm r} 3$.
  ∘ Nominally: $3 \sim_{\rm n} 2$.
• For instance, compare $r = 1$ and $r = 4$ in fig. 4.
  ∘ For $\varepsilon < 0.97$: $\widehat{h}(4,\varepsilon) > \widehat{h}(1,\varepsilon) \implies 4 \succ_{\rm r} 1$.
  ∘ For $\varepsilon > 0.97$: $\widehat{h}(1,\varepsilon) > \widehat{h}(4,\varepsilon) \implies 1 \succ_{\rm r} 4$.
  ∘ Nominally: $4 \sim_{\rm n} 1$.

2.4 Example: Static Poisson Queuing II

¶ Modify example of section 2.3: different uncertainty in probabilities.

¶ Uncertain probability distribution:

• $\widetilde{P}_n$, $n = 0, 1, \ldots$ is the best estimated distribution of number of clients accumulated during the night.

• $\widetilde{P}_n$ may be Poisson with specified average rate $\widetilde{\lambda}$.

• P_n, $n = 0, 1, \ldots$ is the unknown actual distribution of number of clients accumulated during the night.

• The info-gap model for P_n is:

$$\mathcal{U}(h, \widetilde{P}) = \left\{ P_n = \widetilde{P}_n + u_n : \ \max[-\widetilde{P}_n, \ -h\widetilde{P}_n] \le u_n \le h\widetilde{P}_n, \ \sum_{n=0}^{\infty} u_n = 0 \right\}, \quad h \ge 0 \tag{75}$$

¶ Loss function:

• Probability of Not Serving s_2 or more clients is:

$$\pi_{\rm ns}(r, P) = \sum_{n=r+s_2}^{\infty} (\widetilde{P}_n + u_n) \tag{76}$$

• Probability of Unused Capacity for handling s_1 or more clients is:

$$\pi_{\rm uc}(r, P) = \sum_{n=0}^{r-s_1} (\widetilde{P}_n + u_n) \tag{77}$$

• The loss function is:

$$\pi_{\ell}(r, P) = \pi_{\rm uc}(r, P) + \pi_{\rm ns}(r, P) \tag{78}$$

$$= \sum_{n=0}^{r-s_1} (\widetilde{P}_n + u_n) + \sum_{n=r+s_2}^{\infty} (\widetilde{P}_n + u_n) \tag{79}$$

$$= 1 - \sum_{n=r-s_1+1}^{r+s_2-1} (\widetilde{P}_n + u_n) \tag{80}$$

• For instance, if $s_1 = s_2 = 1$:

$$\pi_{\ell}(r, P) = 1 - \widetilde{P}_r - u_r \tag{81}$$

¶ **Performance requirement,** as before in eq.(53), p.14:

$$\pi_\ell(r, P) \le \varepsilon \tag{82}$$

¶ **Robustness** of handling-capacity r to uncertainty in arrival rate λ, as in eq.(54), p.14:

$$\widehat{h}(r, \varepsilon) = \max \left\{ h : \left(\max_{P \in \mathcal{U}(h, \widetilde{P})} \pi_\ell(r, P) \right) \le \varepsilon \right\} \tag{83}$$

¶ **Inner maximum** in eq.(83):

- Suppose $h \le 1$ and $\widetilde{P}_r \le 0.5$.
- Then inner maximum occurs for:

$$u_r = -h\widetilde{P}_r \tag{84}$$

- Denote inner maximum as $M(h)$, as in eq.(55), p.15.
- Thus, from eq.(81) on p.19:

$$M(h) = 1 - \widetilde{P}_r + h\widetilde{P}_r = \varepsilon \tag{85}$$

- Robustness is:

$$\widehat{h}(r, \varepsilon) = \begin{cases} 0 & \text{if } \varepsilon - 1 + \widetilde{P}_r < 0 \\ \dfrac{\varepsilon - 1 + \widetilde{P}_r}{\widetilde{P}_r} & \text{else} \end{cases} \tag{86}$$

¶ **Trade-off** of robustness vs. performance, like eq.(68), p.16:

$$\varepsilon > \varepsilon' \implies \widehat{h}(r, \varepsilon) \ge \widehat{h}(r, \varepsilon') \tag{87}$$

¶ **No robustness of estimated loss,** like eq.(67), p.16:

$$\varepsilon^\star = \pi_\ell(r, \widetilde{P}) = 1 - \widetilde{P}_r \implies \widehat{h}(r, \varepsilon^\star) = 0 \tag{88}$$

¶ **Robustness function,** eq.(86), p.20, and fig. 5:

- $\widehat{h}(r,\varepsilon)$ vs. ε is straight increasing line.
- Two points on the curve are:
  $\widehat{h}(r, 1-\widetilde{P}_r) = 0$.
  $\widehat{h}(r, 1) = 1$.
- Hence:
  - ◦ Robustness curves cross only at maximal robustness.
  - ◦ Nominal preference agrees with robust-satisficing preference.
  - ◦ $\widehat{h}(r,\varepsilon)$ quantifies reliability of sub-optimal performance ($\varepsilon > \varepsilon^\star$).

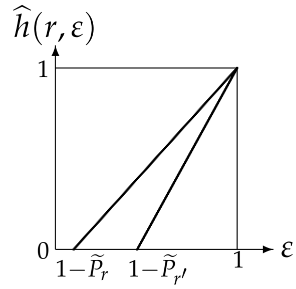

Figure 5: Illustration of robustness curves, eq.(86).

2.5 Example: Dynamic Queuing; Birth and Death Process

¶ **Formulation**

- Server acts while queue is active.
- n = length of queue of clients waiting for service.
- n can be:
  - positive, meaning that clients are waiting for service.
  - negative, meaning that the server is idle.
  - Thus n can be any integer from $-\infty$ to $+\infty$.
  - Note approximation at both extremes.
- $P_n(t)$ = probability that the length is n at time t.

¶ **Birth and death process:** differential equations for $P_n(t)$.

- Client arrivals and "departures" are statistically independent.
- $\lambda \mathrm{d}t$ = probability of 1 client added during $\mathrm{d}t$.
  λ is uncertain.
- $\mu \mathrm{d}t$ = probability of 1 client removed during $\mathrm{d}t$.
  μ is under our control: client-processing rate of server.
- $1 - \lambda \mathrm{d}t - \mu \mathrm{d}t$ = probability of 0 clients added or removed during $\mathrm{d}t$.
- Probability-balance equation for $P_n(t)$:

$$P_n(t + \mathrm{d}t) = P_n(t)(1 - \lambda \mathrm{d}t - \mu \mathrm{d}t) + P_{n-1}(t)\lambda \mathrm{d}t + P_{n+1}(t)\mu \mathrm{d}t + \mathcal{O}(\mathrm{d}t^2) + \cdots \tag{89}$$

- Re-arrange, divide by $\mathrm{d}t$, take limit $\mathrm{d}t \to 0$:

$$\frac{\mathrm{d}P_n(t)}{\mathrm{d}t} = \lambda P_{n-1}(t) - \lambda P_n(t) + \mu P_{n+1}(t) - \mu P_n(t), \quad n \in (-\infty,\ +\infty) \tag{90}$$

- Initial queue size, at $t = 0$, is n_0, so initial conditions for eqs.(90) are:

$$P_n(0) = \delta_{n_0,n} \tag{91}$$

¶ **Moments of $n(t)$:**

$$\mathrm{E}[n^k(t)] = \sum_{n=-\infty}^{\infty} n^k P_n(t) \tag{92}$$

In particular:

$$\overline{n}(t) = \mathrm{E}[n(t)] = \sum_{n=-\infty}^{\infty} n P_n(t) \tag{93}$$

¶ **Moment generating function:**

• Definition:

$$G(z,t) = \sum_n z^n P_n(t) \tag{94}$$

• Derivative:

$$\frac{\partial G(z,t)}{\partial z} = \sum_n n z^{n-1} P_n(t) \tag{95}$$

• Mean queue size:

$$\left. \frac{\partial G(z,t)}{\partial z} \right|_{z=1} = \sum_n n P_n(t) = \mathrm{E}[n(t)] \tag{96}$$

¶ **Deriving** $G(z,t)$:

• Multiply eq.(90), p.22, by z_n and sum on n over $(-\infty, +\infty)$:

$$\begin{aligned}
\sum_n z^n P'_n &= \lambda \sum_n z^n P_{n-1} - (\lambda+\mu) \sum_n z^n P_n + \mu \sum_n z^n P_{n+1} && (97)\\
&= \lambda z \sum_n z^{n-1} P_{n-1} - (\lambda+\mu) \sum_n z^n P_n + \frac{\mu}{z} \sum_n z^{n+1} P_{n+1} && (98)\\
\frac{\partial G(z,t)}{\partial t} &= \lambda z G - (\lambda+\mu) G + \frac{\mu}{z} G && (99)\\
&= \left(\lambda z - (\lambda+\mu) + \frac{\mu}{z} \right) G && (100)
\end{aligned}$$

• Initial condition on $G(z,t)$ at $t=0$, based on eq.(91), p.22:

$$G(z, t=0) = z^{n_0} \tag{101}$$

• Integrate eq.(100) on t:

$$G(z,t) = z^{n_0} \exp\left[\left(\lambda z - (\lambda+\mu) + \frac{\mu}{z} \right) t \right] \tag{102}$$

¶ **Mean queue size:**

Use eqs.(96) and (102) to find:

$$\overline{n}(t,\lambda) = (\lambda-\mu)t + n_0 \tag{103}$$

¶ **Uncertainty in λ:**

$$\mathcal{U}(h,\widetilde{\lambda}) = \left\{\lambda : \max[0,\ (1-h)\widetilde{\lambda}] \le \lambda \le (1+h)\widetilde{\lambda}\right\}, \quad h \ge 0 \tag{104}$$

¶ **Performance requirement:**

$$n_1 \le \overline{n}(t_{\rm c}) \le n_2 \tag{105}$$

- where n_1, n_2 and $t_{\rm c}$ are specified. Typically, $n_1 < 0$ and $n_2 > 0$.
- $t_{\rm c}$ is a clearing time chosen by the designer.
- Denote the performance specification $s = (n_1, n_2)$.
- Denote the design variables $q = (\mu, t_{\rm c})$.

¶ **Robustness** with design variables q and specifications s:

$$\widehat{h}(q,s) = \max\left\{h : \left(\max_{\lambda\in\mathcal{U}(h,\widetilde{\lambda})} \overline{n}(t_{\rm c},\lambda)\right) \le n_2 \ \text{ and } \ \left(\min_{\lambda\in\mathcal{U}(h,\widetilde{\lambda})} \overline{n}(t_{\rm c},\lambda)\right) \ge n_1\right\} \tag{106}$$

¶ **Sub-problem robustnesses:**

$$\widehat{h}_1(q,s) = \max\left\{h : \left(\min_{\lambda\in\mathcal{U}(h,\widetilde{\lambda})} \overline{n}(t_{\rm c},\lambda)\right) \ge n_1\right\} \tag{107}$$

$$\widehat{h}_2(q,s) = \max\left\{h : \left(\max_{\lambda\in\mathcal{U}(h,\widetilde{\lambda})} \overline{n}(t_{\rm c},\lambda)\right) \le n_2\right\} \tag{108}$$

Since both requirements are necessary:

$$\widehat{h}(q,s) = \min[\widehat{h}_1(q,s),\ \widehat{h}_2(q,s)] \tag{109}$$

¶ **Deriving $\widehat{h}_2$:**

$$\max_{\lambda\in\mathcal{U}(h,\widetilde{\lambda})} [(\lambda-\mu)t_{\rm c} + n_0] \le n_2 \implies \left[(1+h)\widetilde{\lambda} - \mu\right] t_{\rm c} + n_0 \le n_2 \tag{110}$$

Thus:

$$\widehat{h}_2(q,s) = \begin{cases} \dfrac{n_2 - n_0}{\widetilde{\lambda} t_{\rm c}} + \dfrac{\mu}{\widetilde{\lambda}} - 1 & \text{if } (\widetilde{\lambda}-\mu)t_{\rm c} + n_0 \le n_2 \\ 0 & \text{else} \end{cases} \tag{111}$$

¶ **Deriving $\widehat{h}_1$:**

- The inner minimum in eq.(107) is a decreasing function of h (fig. 6):

$$\min_{\lambda \in \mathcal{U}(h,\widetilde{\lambda})} \overline{n}(t_c, \lambda) = \begin{cases} \left[(1-h)\widetilde{\lambda} - \mu\right] t_c + n_0 & \text{if } h \leq 1 \\ -\mu t_c + n_0 & \text{else} \end{cases} \tag{112}$$

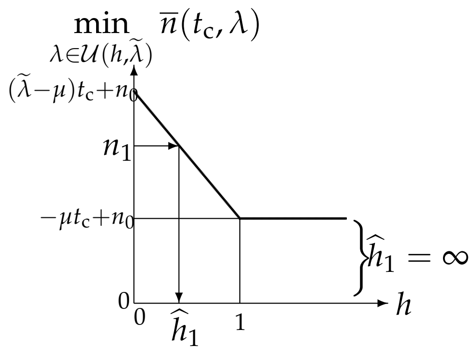

Figure 6: Schematic illustration of the evaluation of $\widehat{h}_1$ from eq.(112).

- Thus:

$$\widehat{h}_1(q,s) = \begin{cases} 0 & \text{if } (\widetilde{\lambda} - \mu)t_c + n_0 \leq n_1 \\ 1 - \dfrac{n_1 - n_0}{\widetilde{\lambda} t_c} - \dfrac{\mu}{\widetilde{\lambda}} & \text{if } -\mu t_c + n_0 \leq n_1 < (\widetilde{\lambda} - \mu)t_c + n_0 \\ \infty & \text{if } n_1 < -\mu t_c + n_0 \end{cases} \tag{113}$$

¶ $\widehat{h}(q,s)$ from combining eqs.(109), (111) and (113).

¶ Maximal robustness.

• From eq.(109), p.24, we see that the choice of $q = (\mu, t_c)$ which maximizes $\widehat{h}(q,s)$ is the choice which causes:

$$\widehat{h}_1(q,s) = \widehat{h}_2(q,s) \tag{114}$$

• Suppose that n_1 and n_2 are such that $\widehat{h}_1(q,s)$ and $\widehat{h}_2(q,s)$ are both positive and finite.

• Then eq.(114) is:

$$1 - \frac{n_1 - n_0}{\widetilde{\lambda} t_c} - \frac{\mu}{\widetilde{\lambda}} = \frac{n_2 - n_0}{\widetilde{\lambda} t_c} + \frac{\mu}{\widetilde{\lambda}} - 1 \tag{115}$$

which implies:

$$\mu = \widetilde{\lambda} + \frac{\Delta}{t_c} \quad \text{where} \quad \Delta = n_0 - \frac{n_1 + n_2}{2} \tag{116}$$

• That is, for any t_c, choosing μ according to eq.(116) maximizes $\widehat{h}(q,s)$ for that t_c.

• For any t_c, the robustness with μ from eq.(116) is:

$$\widehat{h}(q,s) = \widehat{h}_1(q,s) = \widehat{h}_2(q,s) = \frac{n_2 - n_1}{2\widetilde{\lambda} t_c} \tag{117}$$

provided that n_1 and n_2 are such that $\widehat{h}_1(q,s)$ and $\widehat{h}_2(q,s)$ are both positive and finite.

• We see from eq.(117) the following trade-offs:

◦ Robustness increases as acceptable un-used capacity increases (as n_1 becomes more negative):

$$\frac{\partial \widehat{h}(q,s)}{\partial n_1} < 0 \tag{118}$$

◦ Robustness increases as the acceptable # of un-served clients increases:

$$\frac{\partial \widehat{h}(q,s)}{\partial n_2} > 0 \tag{119}$$

◦ Robustness increases as the tolerance-window $n_2 - n_1$ increases:

$$\frac{\partial \widehat{h}(q,s)}{\partial (n_2 - n_1)} > 0 \tag{120}$$

◦ Robustness increases as clearing time decreases:

$$\frac{\partial \widehat{h}(q,s)}{\partial t_c} < 0 \tag{121}$$

3 Probabilistic Info-Gap Parameter

¶ Basic idea:
- ○ Complex temporal or spatial waveforms are modelled by an info-gap model, $\mathcal{U}(h, \widetilde{u})$, $h \geq 0$.
- ○ The uncertainty parameter h has physical meaning. E.g. energy of event.
- ○ The uncertainty in h is represented by a pdf.

¶ Example:
- ○ Dynamic system with uncertain load $u \in \mathcal{U}(h, \widetilde{u}),\ h \geq 0$.
- ○ Load u causes damage $\delta(u)$.
- ○ Failure if:

$$\delta_u(t) \geq \Delta_c \tag{122}$$

¶ Robustness:

$$\widehat{h}(q, \Delta_c) = \max \left\{ h : \left(\max_{u \in \mathcal{U}(h, \widetilde{u})} \delta_u(t) \right) \leq \Delta_c \right\} \tag{123}$$

q is the vector of decision variables.

¶ Failure **can not occur** if:

$$h < \widehat{h}(q, \Delta_c) \tag{124}$$

¶ Failure **need not occur** even if:

$$h \geq \widehat{h}(q, \Delta_c) \tag{125}$$

(Load may be propitious.)

¶ We **cannot calculate** P_f because $p(u)$ is unknown.
¶ We **can calculate** an upper bound for P_f:

$$P_f \leq \text{Prob}\left[h \geq \widehat{h}(q, \Delta_c) \right] = 1 - P\left[\widehat{h}(q, \Delta_c) \right] \tag{126}$$

$P(\cdot)$ is the cumulative probability distribution of h.

¶ Optimal q:
- ○ We can seek q to maximize $\widehat{h}(q, \Delta_c)$.
- ○ $P(h)$ is a monotonically increasing function.
- ○ Thus maximizing $\widehat{h}(q, \Delta_c)$ also maximizes $P(\widehat{h})$ and minimizes $1 - P(\widehat{h})$.

¶ Proof:

$$\partial P(h)/\partial h \geq 0 \tag{127}$$

and because:

$$\frac{\partial P\left[\widehat{h}(q,\Delta_{\rm c})\right]}{\partial q} = \frac{\partial P\left[\widehat{h}(q,\Delta_{\rm c})\right]}{\partial h}\frac{\partial \widehat{h}(q,\Delta_{\rm c})}{\partial q} \tag{128}$$

QED

¶ Equivalent definition of the robust optimal action $\widehat{q}$:

$$\widehat{h}(\widehat{q},\Delta_{\rm c}) = \max_{q\in\mathcal{Q}} P\left[\widehat{h}(q,\Delta_{\rm c})\right] \tag{129}$$

¶ Likewise, $P(\cdot)$ defines the same preference ordering on q as $\widehat{h}(q,\Delta_{\rm c})$:

$$q \succ q' \quad \text{if} \quad P\left[\widehat{h}(q,\Delta_{\rm c})\right] \; > \; P\left[\widehat{h}(q',\Delta_{\rm c})\right] \tag{130}$$

¶ This provides a probabilistic calibration of the relative merits of the options.