Lecture Notes on
Two-Period Investments and the Equity Premium Puzzle

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Source material: Yakov Ben-Haim, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd ed., Academic Press, section 11.5.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

Contents

1 What is the Equity Premium Puzzle?

¶ Sources:

• Cochrane, John H., 2001, *Asset Pricing,* Princeton University Press, Princeton, chap. 1.

• Kocherlakota, Narayana R., 1996, The equity premium: It's still a puzzle, *Journal of Economic Literature,* 34: 42–71.

• Mehra, Rajnish and Edward C. Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics,* 15 (March) pp.145–161.

¶ The equity premium puzzle:

• The average return to risky assets such as stocks over the past century has been about 8%.

• The average return to essentially riskless investments, such as US Treasury bonds, over the past century has been about 1%.

• The "equity premium" is the excess return to risky assets over riskless assets. So the equity premium over the past century has been about 7%.

• It makes sense that, at equilibrium, risky assets return more than riskless assets. Otherwise nobody would invest in risky assets.

• Why is the equity premium 7% and not 3% or 11%?

• The economists have developed a large collection of quantitative models for predicting and explaining the magnitude of the equity premium. The major class of models is called "capital asset pricing models" (CAPM).

• These models do not work very well.

¶ **Utility functions.**

- c = quantity of consumption (of money, popcorn, etc.)
- $u(c)$ = utility from consumption c.
- Economists typically assume that:

$$\frac{\mathrm{d}u(c)}{\mathrm{d}c} > 0 \tag{1}$$

Meaning: more consumption is better than less.

- What can/should we say about the 2nd derivative:

$$\frac{\mathrm{d}^2u(c)}{\mathrm{d}c^2} \ ? \ 0 \tag{2}$$

What is the meaning of the 2nd derivative?
It expresses aversion to risk, as we now explain.

¶ **Gambles and the sure thing.**

- Consider low, average and high consumption:

$$c_1 \ < \ \bar{c} = \frac{c_1 + c_2}{2} \ < \ c_2 \tag{3}$$

Thus:

$$u(c_1) < u(\bar{c}) < u(c_2) \tag{4}$$

- Would you prefer to consume:
  - Option 1: $\bar{c}$ for sure, or,
  - Option 2: either c_1 or c_2 with 0.5 probability for each?
- Attitude to risk:
  - Prefer option 1: risk averse.
  - Prefer option 2: risk loving.
- The expected utility of these options are:
  - Option 1: $u(\bar{c})$.
  - Option 2: $0.5u(c_1) + 0.5u(c_2)$.

¶ **Arrow-Pratt risk aversion:**

• A prediction of expected utility theory is:

$$\text{Prefer option 1 if:} \quad u(\bar{c}) > 0.5u(c_1) + 0.5u(c_2) \tag{5}$$

$$\text{Prefer option 2 if:} \quad u(\bar{c}) < 0.5u(c_1) + 0.5u(c_2) \tag{6}$$

$$\text{Indifferent if:} \quad u(\bar{c}) = 0.5u(c_1) + 0.5u(c_2) \tag{7}$$

• Curvature of utility function expresses risk aversion of proclivity, as in figs. 1 and 2.

• The dimensionless Arrow-Pratt risk aversion coefficient is:

$$\gamma = -\frac{u''}{(u')^2} \tag{8}$$

• Typical values of Arrow-Pratt risk aversion: γ from 0.5 to 3.0.

• CAPM models include the concept of Arrow-Pratt risk aversion.

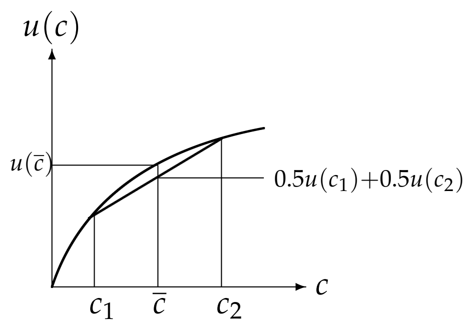

Figure 1: Utility function for risk-averse consumption, eq.(5).

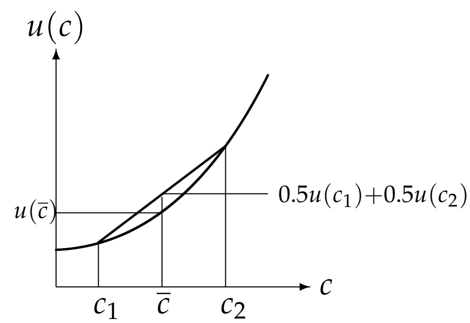

Figure 2: Utility function for risk-loving consumption, eq.(6).

¶ **What is the equity premium puzzle?**

• The equity premium is positive because investors are risk averse.

• CAPMs explain observed equity premium ($\sim$ 7%) with $\gamma \approx 50$.

• Typical values of Arrow-Pratt risk aversion: γ from 0.5 to 3.0.

• $\rightarrow\leftarrow$

2 Two-Period Single-Asset Investment

¶ **2-period, single-asset investment problem:**
ξ = amount of asset purchased in first period. $\xi \geq 0$.
Decision: choose ξ.
y_2 = current asset price plus dividend in 2nd period per share.
$y_2\xi$ = payoff in 2nd period: current price plus dividend from investment.
The problem: y_2 unknown when ξ must be chosen.

¶ **What is known:**
p_1 = current asset price.
e_1 = endowment in first period.
e_2 = endowment in second period.
$c_1 = e_1 - p_1\xi$ = consumption in first period.
$c_2 = e_2 + y_2\xi$ = consumption in second period.
$u(c_1)$ = first-period utility from consumption c_1.
$u(c_2)$ = second-period utility from consumption c_2.
$\dfrac{\mathrm{d}u(c)}{\mathrm{d}u} > 0$: utility increases with consumption.
Usually assume: $\dfrac{\mathrm{d}^2u(c)}{\mathrm{d}u^2} < 0$: decreasing marginal utility.
β = subjective discount factor. $\beta < 1$.
$\beta u(c_2)$ = first-period utility from consumption c_2 in 2nd period.
$U(y_2, \xi)$ = total discounted first-period utility:

$$U(y_2,\xi) = u(c_1) + \beta u(c_2) \tag{9}$$
$$= u(\underbrace{e_1 - p_1\xi}_{c_1}) + \beta u(\underbrace{e_2 + y_2\xi}_{c_2}) \tag{10}$$

¶ **Expected-utility maximization:**

Choose ξ to:

- Maximize the expected discounted utility $U(y_2, \xi)$,
- Satisfy budget constraints on consumptions c_1 and c_2:
  No borrowing, so $c_1 \geq 0$ and $c_2 \geq 0$.

$$\xi^{\star} = \arg\max_{\xi} \mathrm{E}_1[U(y_2, \xi)] \tag{11}$$

$\mathrm{E}_1(\cdot)$ is expectation based on first-period information.

2.1 Info-gap Robust Satisficing

¶ We now develop an **info-gap robust-satisficing** approach. The main conceptual modification is that the utility will be satisficed (not maximized) and the robustness to uncertainty will be maximized.

¶ **Robustness question:** how wrong can the current estimate of future returns be, without jeopardizing the attainment of a specified level of utility?
¶ **Preferences on options** come from the answer to the robustness question.
¶ No probabilistic information required.

¶ **Info-gap model** for uncertainty in future returns:
$\widetilde{y}_2$ = best estimate of future payoff.
y_2 = unknown actual future payoff.
Fractional error of $\widetilde{y}_2$ unknown:

$$\mathcal{Y}(\alpha, \widetilde{y}_2) = \{y_2 : \ y_2 \geq 0, \ |y_2 - \widetilde{y}_2| \leq \alpha \widetilde{y}_2\}, \quad \alpha \geq 0 \tag{12}$$

¶ $\mathcal{Y}(\alpha, \widetilde{y}_2)$, $\alpha \geq 0$: unbounded family of nested sets of payoffs:

¶ **Critical utility:**
- Investor prefers more utility rather than less.
- $\overline{U}$ is the lowest acceptable discounted 2-period utility.
- $\overline{U}$ is a **reservation utility.**
- If $\overline{U}$ cannot be reasonably anticipated then reject investment.

¶ **Robustness** of investment ξ given aspiration $\overline{U}$:

$$\widehat{\alpha}(\xi, \overline{U}) = \max \left\{ \alpha : \min_{y_2 \in \mathcal{Y}(\alpha, \widetilde{y}_2)} U(y_2, \xi) \geq \overline{U} \right\} \tag{13}$$

¶ **Robust satisficing:**
- More robustness is preferable to less, at same aspiration $\overline{U}$.
- Info-gap robust-satisficing allocation:
  - Maximize robustness.
  - Satisfice aspiration.
  - Obey budget constraints on consumptions c_1 and c_2:

$$\widehat{\xi}(\overline{U}) = \arg\max_{\xi} \widehat{\alpha}(\xi, \overline{U}) \tag{14}$$

¶ **Finding** $\min U(y_2, \xi)$ in eq.(13):
- $u(c)$ increases monotonically in consumption c.
- Hence $\min U(y_2, \xi)$, at horizon of uncertainty α, occurs when y_2 takes its minimal value: $y_2 = (1-\alpha)\widetilde{y}_2$ provided $\alpha \le 1$, or $y_2 = 0$ otherwise.

Let:

$$\mu_s(\alpha) = \min_{y_2 \in \mathcal{Y}(\alpha, \widetilde{y}_2)} U(y_2, \xi) \tag{15}$$

which can be expressed explicitly as:

$$\mu_s(\alpha) = \begin{cases} u(e_1 - p_1\xi) + \beta u[e_2 + (1-\alpha)\widetilde{y}_2\xi] & \text{if } \alpha \le 1 \\ u(e_1 - p_1\xi) + \beta u(e_2) & \text{else} \end{cases} \tag{16}$$

¶ **Computing the robustness.**
- $\mu_s(\alpha)$ is plotted schematically in fig. 3.
- At any value of $\overline{U}$, the robustness $\widehat{\alpha}$ is
  the greatest value of α for which $\mu_s(\alpha) \ge \overline{U}$.
- Fig. 3 is $\widehat{\alpha}(\xi, \overline{U})$ on horizontal axis and $\overline{U}$ on vertical axis.

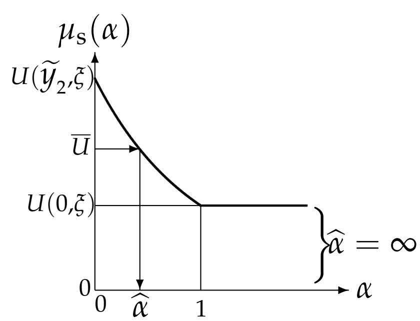

Figure 3: Schematic illustration of the evaluation of the robustness from the function $\mu_s(\alpha)$ in eq.(16).

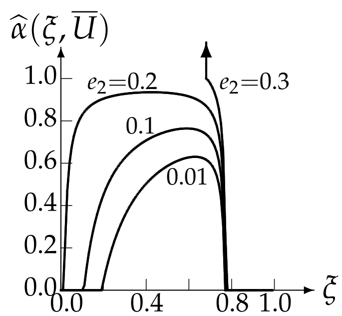

Figure 4: Robustness vs. allocation. $e_1 = 1$, $p_1 = 1$, $\gamma = 3$, $\beta = 0.9$, $\widetilde{y}_2 = 1.1$, $\overline{U} = -10$.

¶ **Example: constant risk aversion utility function:**

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } c \geq 0 \\ -\infty & \text{else} \end{cases} \tag{17}$$

With this utility function and for $\gamma \neq 1$, the robustness of allocation ξ with aspiration $\overline{U}$ is:

$$\widehat{\alpha}(\xi, \overline{U}) = \begin{cases} 0 & \text{if } U(\widetilde{y}_2, \xi) \leq \overline{U} \\ 1 + \frac{1}{\widetilde{y}_2 \xi} \left[e_2 - \left(\frac{[\overline{U} - u(e_1 - p_1 \xi)](1-\gamma)}{\beta} \right)^{\frac{1}{1-\gamma}} \right] & \text{if } U(0, \xi) \leq \overline{U} < U(\widetilde{y}_2, \xi) \\ \infty & \text{if } \overline{U} < U(0, \xi) \end{cases} \tag{18}$$

¶ The robustness function is plotted in fig. 4 versus the quantity of asset purchased, ξ, for several values of the endowment in the second period, e_2. The initial endowment is $e_1 = 1$, the asset price is $p_1 = 1$, the anticipated payoff is $\widetilde{y}_2 = 1.1$, the discount factor is $\beta = 0.9$ and the exponent of the utility function is $\gamma = 3$. The reservation utility is $\overline{U} = -10$.

¶ The general shape of the robustness curves is an inverted 'U'. If the investment ξ is too small or too high then the anticipated utility, $U(\widetilde{y}_2, \xi)$, is less than the reservation utility $\overline{U}$ so the robustness is zero (first line of eq.(18)).

¶ The robustness increases as the 2nd-period endowment increases since $\overline{U}$ becomes easier to obtain. Thus e_2 increases from the lower to the upper curves.

¶ When $e_2 = 0.3$ the robustness is infinite for $\xi \leq 0.68$ since the reservation utility is obtained even with no return on the assets (third line of eq.(18)).

2.2 Trade-off of Robustness Against Utility

¶ **Trade-off of Robustness Against Utility:**

- As aspired utility $\overline{U}$ increases,
  the immunity against uncertainty $\widehat{\alpha}(\xi, \overline{U})$ decreases.
- This theorem holds at:
  - Any fixed value of investment ξ.
  - For the robust-satisficing investment $\widehat{\xi}(\overline{U})$.
- Thus:
  - $\widehat{\alpha}(\xi, \overline{U})$ decreases monotonically as $\overline{U}$ increases, as in fig. 3, p.10.
  - $\widehat{\alpha}(\widehat{\xi}(\overline{U}), \overline{U})$ decreases monotonically as $\overline{U}$ increases.
- Formally:

$$\overline{U} > \overline{U}' \quad \text{implies} \quad \widehat{\alpha}(\xi, \overline{U}) \le \widehat{\alpha}(\xi, \overline{U}') \tag{19}$$

$$\text{and} \quad \widehat{\alpha}(\widehat{\xi}(\overline{U}), \overline{U}) \le \widehat{\alpha}(\widehat{\xi}(\overline{U}'), \overline{U}') \tag{20}$$

¶ **Anticipated utility has zero robustness:**

- $\widetilde{y}_2$ = anticipated return in 2nd period.
- $U(\widetilde{y}_2, \xi)$ = anticipated utility from investment ξ.

$$\overline{U} = U(\widetilde{y}_2, \xi) \quad \text{implies} \quad \widehat{\alpha}(\xi, \overline{U}) = 0 \tag{21}$$

- No investment can be relied upon to result in the anticipated utility.
- Only aspirations $\overline{U} < U(\widetilde{y}_2, \xi)$ have positive robustness.
- Note that, in principle:

$$U(\widetilde{y}_2, \xi) \ne \mathrm{E}_1[U(y_2, \xi)] \tag{22}$$

though the difference may be small.

¶ Relations (19)–(21) are illustrated in fig. 5, p.13.

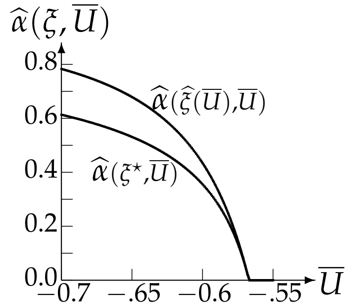

Figure 5: Robustness–utility trade-off, illustrating eqs.(19) and (20).

¶ Details of calculation of fig. 5.

• To calculate the expected utility based on first-period information, $E_1(U)$, we assume that y_2 is lognormally distributed with density:

$$p(y_2) = \frac{1}{\sigma y_2 \sqrt{2\pi}} \exp\left[-(\ln y_2 - \mu)^2/(2\sigma^2)\right], \quad y_2 > 0 \tag{23}$$

with $\mu = 0.0676587 = \ln \widetilde{y}_2$ for $\widetilde{y}_2 = 1.07$, and $\sigma = 0.131192$.

• Note that $E(y_2) = \exp[\mu + (\sigma^2/2)] \approx \exp(\mu)$.

• The maximum expected utility is

$$\max_{\xi} E_1(y_2, \xi) = -0.568 \tag{24}$$

which results when the investment is

$$\xi^\star = 4.15 \tag{25}$$

This value of maximum expected utility is very close to the utility $U(\widetilde{y}_2, \xi^\star)$ of the anticipated return $\widetilde{y}_2$. Thus the inequality (22) is very nearly an equality

$$U(\widetilde{y}_2, \xi^\star) \approx E_1[U(y_2, \xi^\star)] \tag{26}$$

¶ Fig. 5 illustrates:

- **Trade-off** between:
  robustness-against-uncertainty and
  aspiration for utility, eqs.(19) and (20).
- **Zero robustness** at nominal aspiration, eq.(21).

¶ **Comparing robust-satisficing and utility-maximizing:**

- The robust-satisficing investment is $\widehat{\xi}(\overline{U})$.
- The utility-maximizing investment is $\xi^{\star}$.

- The utility-maximizing aspiration is:
  $\max_{\xi} \mathrm{E}_1[U(y_2, \xi)] = \mathrm{E}_1[U(y_2, \xi^{\star})] = -0.568$.

- When $\overline{U} = \max_{\xi} \mathrm{E}_1[U(y_2, \xi)] = -0.568$
  Then $\widehat{\xi}(\overline{U}) \approx \xi^{\star}$.

- From eq.(26), p. 13, we know: $U(\widetilde{y}_2, \xi^{\star}) \approx \mathrm{E}_1[U(y_2, \xi^{\star})]$.
- Thus from eq.(21), p. 12:
  $\overline{U} = \max_{\xi} \mathrm{E}_1[U(y_2, \xi)]$ implies $\widehat{\alpha}(\widehat{\xi}(\overline{U}), \overline{U}) \approx \widehat{\alpha}(\xi^{\star}, \overline{U}) \approx 0$.
  That is: maximum expected utility (MEU) has zero robustness to uncertainty.
- Furthermore, by definition and eqs.(19) and (20), p. 12:
  $\overline{U} < \max_{\xi} \mathrm{E}_1[U(y_2, \xi)]$ implies $\widehat{\alpha}(\widehat{\xi}(\overline{U}), \overline{U}) > \widehat{\alpha}(\xi^{\star}, \overline{U}) > 0$.
  That is: **robust-satisficing strategy is more robust than MEU strategy at all positive robustness.**

¶ **Comparing robust-satisficing and utility-maximizing,** continued:

- For example:
  - $\overline{U} = -0.6$ implies:
    Maximizing expected utility: $\widehat{\alpha}(\zeta^\star, \overline{U}) = 0.36$
    Robust-satisficing: $\widehat{\alpha}(\widehat{\zeta}, \overline{U}) = 0.43$.
- For example:
  - $\overline{U} = -0.65$ implies:
    Maximizing expected utility: $\widehat{\alpha}(\zeta^\star, \overline{U}) = 0.53$
    Robust-satisficing: $\widehat{\alpha}(\widehat{\zeta}, \overline{U}) = 0.68$.
- "Robustness premium" for robust-satisficing with $\widehat{\zeta}(\overline{U})$
  rather than utility-maximizing with $\zeta^\star$
  is more than 20% in both cases.

¶ **Plausible behavioral response:**
Investor robust-satisfices rather than maximizes utility.

2.3 Realized Utility

¶ **The question:**

ξ = investment.
y_2 = actual payoff.
$U(\xi, y_2)$ = realized utility.
$\xi^\star$ = investment which maximizes expected utility (EU), based on $\widetilde{y}_2$.
$\widehat{\xi}(\overline{U})$ = investment which maximizes robustness given utility-aspiration $\overline{U}$,
based on $\widetilde{y}_2$.
$U(\xi^\star, y_2)$ = utility actually realized with EU-maximizing strategy.
$U(\widehat{\xi}(\overline{U}), y_2)$ = utility actually realized with robust-satisficing strategy.
The question: when is $U(\xi^\star, y_2) > U(\widehat{\xi}(\overline{U}), y_2)$?

$$\Delta(y_2) = U(y_2, \xi^\star) - U(y_2, \widehat{\xi}(\overline{U})) \tag{27}$$

For what y_2 is the differential utility $\Delta(y_2)$, positive,
in favor of maximizing EU?
$\Delta(y_2) > 0$ implies EU-maximizing strategy performs better.
$\Delta(y_2) < 0$ implies robust-satisficing strategy performs better.

¶ **Details of the computations:**

• We assume that y_2 is lognormally distributed with mean and standard deviation 0.0676587 and 0.131192. This mean equals $\ln \widetilde{y}_2$ for $\widetilde{y}_2 = 1.07$.

• We use the same utility function as before, eq.(17), p. 11, with $\gamma = 3$.

• The discount factor is $\beta = 0.9$.

• The initial endowment and asset price are $e_1 = 1$ and $p_1 = 1$. We will consider a range of values of the 2nd-period endowment e_2.

• The utility aspiration for the robust-satisficing calculation is evaluated as a constant fraction of the maximum expected utility:

$$\overline{U} = \begin{cases} \phi \max_{\xi} \mathrm{E}_1[U(y_2, \xi^\star)] & \text{if } \max_{\xi} \mathrm{E}_1[U(y_2, \xi^\star)] \leq 0 \\ \max_{\xi} \mathrm{E}_1[U(y_2, \xi^\star)]/\phi & \text{else} \end{cases} \tag{28}$$

$\phi = 1.06$ so the satisficing aspiration $\overline{U}$ is for utility which is no lower than 6% less than the maximum expected utility.

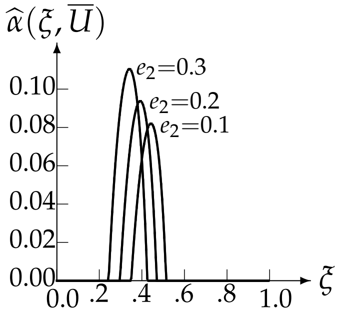

Figure 6: Robustness functions. $p_1 = e_1 = 1$, $\gamma = 3$, $\beta = 0.9$, $\phi = 1.06$, $\widetilde{y}_2 = 1.07$.

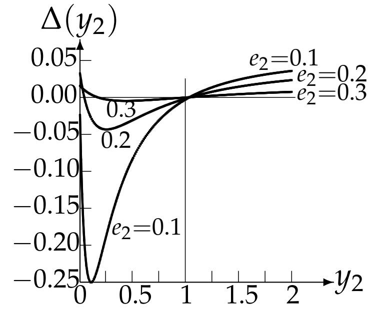

Figure 7: Realized utility difference, $\Delta(y_2)$.

¶ **Robustness: fig. 6:**

• Robustness functions $\widehat{\alpha}(\xi,\overline{U})$ vs. investment ξ are shown in fig. 6 for three values of the 2nd-period endowment e_2.

• The maximum EU for $e_2 = 0.1$, 0.2 and 0.3 are $U(\widetilde{y}_2,\xi^\star) = -3.00$, -2.54 and -2.17 respectively, improving as the endowment increases.

• The corresponding utility aspirations, eq.(28), p. 17, with $\phi = 1.06$, are $\overline{U} = -3.18$, -2.69 and -2.30.

• The robustness functions are quite narrow, with a sharply defined robust-satisficing investment which maximizes the robustness.

• The robust-satisficing investment diminishes as the 2nd-period endowment increases: less investment is required in order to satisfice the utility and maximize the robustness.

• The robust-satisficing investments, for $e_2 = 0.1$, 0.2 and 0.3, are $\widehat{\xi}(\overline{U}) = 0.441$, 0.391 and 0.341 respectively.

• Likewise, the utility-maximizing investments diminish as e_2 increases: for $e_2 = 0.1$, 0.2 and 0.3 they are $\xi^\star = 0.431$, 0.381 and 0.336 respectively. Less investment is needed in order to maximize utility as the endowment rises.

• We note that the robust-satisficing investment slightly exceeds the utility-maximizing investment in each case.

¶ **Realized utility difference,** $\Delta(y_2)$:

- $\Delta(y_2)$ is shown in fig. 7, p. 18, for three values of the 2nd-period endowment.
- $\Delta(y_2) > 0$ for $y_2 \geq 1.04$ when $e_2 = 0.1$ and 0.2, and for $y_2 \geq 1.01$ when $e_2 = 0.3$.
- $\Delta(y_2) < 0$ elsewhere, except for very small values of y_2 when $e_2 = 0.1$ and 0.2.
- Recall that the anticipated return was $\widetilde{y}_2 = 1.07$. In other words:
  - ◦ The expected-utility maximizing strategy $\xi^\star$ performs better than the robust-satisficing strategy $\widehat{\xi}$ when the returns are sufficiently large.
  - ◦ Robust-satisficing (almost always) performs better otherwise.

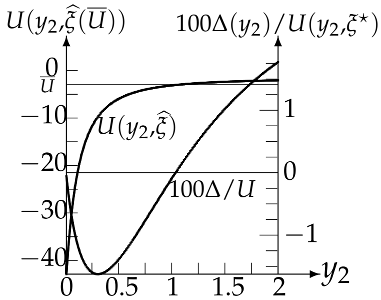

Figure 8: Utility (U) and relative utility premium ($100\Delta/U$). $p_1 = e_1 = 1$, $e_2 = 0.1$, $\gamma = 3$, $\beta = 0.9$, $\phi = 1.06$, $\widetilde{y}_2 = 1.07$, $\overline{U} = -3.183$.

¶ **Fractional utility difference:** fig. 8.
Typically: $\Delta(y_2) \leq 0.02U(y_2,\xi^\star)$.

¶ **Behavioral-Psychological Question:**

• What type of investor would tend to prefer the EU-maximizing strategy $\zeta^\star$ and what type would tend to prefer the robust-satisficing strategy?

• The maximizing strategy is dominant, in terms of realized return, when the actual return is no less than 3 to 6 percentage points below the anticipated return.

• The robust-satisficing strategy is dominant otherwise.

• An **"optimistic"** investor is one who anticipates that actual realized returns y_2 will be better than the anticipated return $\widetilde{y}_2$.

• A **"pessimist"** will expect lower returns than the anticipation.

• The **optimist** thinks that $\widetilde{y}_2$ is an **under** estimate.

• The **pessimist** thinks that $\widetilde{y}_2$ is an **over** estimate.

• The **optimist** is one whose outlook emphasizes the favorable possibilities.[1]

• The **pessimist** is concerned primarily with avoiding catastrophes.

• Fig. 7 suggests that:

  ◦ The **optimist** will tend to prefer the EU-maximizing strategy.

  ◦ The **pessimist** will tend to robust-satisfice.

[1] This suggests that the optimist may wish to opportunely windfall rather than to robust-satisficing. We have not considered this possibility yet. We will later.

3 Robust-Satisficing, Opportune-Windfalling and Maximizing: Four Theorems

• In this section we present four theorems which underlie our analysis of robustness, opportunity, satisficing, windfalling and maximizing.
• In section 3.1 we define the **info-gap robust-satisficing,** and in section 3.2 we present two theorems.
• In section 3.3 we define **info-gap opportune windfalling,** and in section 3.4 we present two more theorems.

3.1 Info-gap Robust-Satisficing

¶ Info-gap models of uncertainty.
$\Re$ = non-negative reals.
S = a Banach space of uncertain vectors or functions u.
An info-gap model is a set-valued function from $\Re \times S$ into the power set of S.
An info-gap model obeys two axioms.
Nesting:

$$\alpha < \alpha' \quad \text{implies} \quad \mathcal{U}(\alpha, \widetilde{u}) \subset \mathcal{U}(\alpha', \widetilde{u}) \tag{29}$$

Contraction:

$$\mathcal{U}(0, \widetilde{u}) = \{\widetilde{u}\} \tag{30}$$

¶ **Decision and Reward:**
$q =$ the **decision vector** whose elements can be linguistic variables as well as real valued variables or functions.
$R(q, u) =$ **reward function:** a real valued function for which large values are more desirable than small values.
$R(q, u)$ depends on both the decision q and the uncertain variables u whose uncertainty is described by the info-gap model $\mathcal{U}(\alpha, \widetilde{u})$.
The reward function is **lower unsatiated** if:

$$\alpha < \alpha' \quad \text{implies} \quad \min_{u \in \mathcal{U}(\alpha, \widetilde{u})} R(q, u) > \min_{u \in \mathcal{U}(\alpha', \widetilde{u})} R(q, u) \tag{31}$$

• Lower unsatiation is a type of monotonicity: the worst reward, up to uncertainty α, gets strictly worse as the horizon of uncertainty α increases.
• This monotonicity in α does not imply monotonicity of $R(q, u)$ in either q or u. Lower unsatiation results from the nesting of the sets in the info-gap model $\mathcal{U}(\alpha, \widetilde{u})$.

¶ **Robustness of decision q:**
The **robustness** of decision q, with aspiration that the reward be no less than r_c, is the greatest horizon of uncertainty up to which the aspiration is guaranteed:

$$\widehat{\alpha}(q, r_c) = \max \left\{ \alpha : \min_{u \in \mathcal{U}(\alpha, \widetilde{u})} R(q, u) \geq r_c \right\} \tag{32}$$

$\widehat{\alpha}(q, r_c) = 0$ by definition if the set in eq.(32) is empty for all values of α. This occurs if and only if $R(q, \widetilde{u}) < r_c$.

¶ **Decision strategies:**

• A **maximizing** decision maximizes the reward function, evaluated at the nominal value of u, subject to possible constraints on the choice of q:

$$q^{\star} = \arg\max_{q} R(q, \widetilde{u}) \tag{33}$$

• A **robust-satisficing** decision at aspiration r_{c} maximizes the robustness, subject to possible constraints on the choice of q:

$$\widehat{q}_{\mathrm{s}}(r_{\mathrm{c}}) = \arg\max_{q} \widehat{\alpha}(q, r_{\mathrm{c}}) \tag{34}$$

• **Non-uniqueness:** It can happen that the robust-satisficing decision is not unique, in which case we need a rule for choosing $\widehat{q}_{\mathrm{s}}$. The only situation of concern to the current discussion in which this happens is when the maximum robustness equals zero: $\widehat{\alpha}(\widehat{q}, r_{\mathrm{c}}) = 0$. In this case we define the robust-satisficing decision to be equal to the maximizing decision:

$$\widehat{q}_{\mathrm{s}}(r_{\mathrm{c}}) = q^{\star} \quad \text{if} \quad \widehat{\alpha}(\widehat{q}_{\mathrm{s}}(r_{\mathrm{c}}), r_{\mathrm{c}}) = 0 \tag{35}$$

3.2 Two Theorems

¶ Robustness trades-off against aspiration for reward:

Theorem 1 *$\mathcal{U}(\alpha, \widetilde{u})$ is an info-gap model, $R(q, u)$ is a lower unsatiated reward function, and $\widehat{\alpha}(q, r_c)$ is the corresponding robustness function. Let r be an aspiration whose robustness is positive for decision vector q: $\widehat{\alpha}(q, r) > 0$. Any greater aspiration with q has lower robustness:*

$$r' > r \quad \textit{implies} \quad \widehat{\alpha}(q, r') < \widehat{\alpha}(q, r) \tag{36}$$

¶ Robustness Strategy and Reliable Reward:
If q is a more robust decision than q', at reward-aspiration r_c, then the lowest reward with q, up to horizon of uncertainty equal to the robustness of q, is greater than the lowest reward from q' up to the same uncertainty.

Theorem 2 *$\mathcal{U}(\alpha, \widetilde{u})$ is an info-gap model, $R(q, u)$ is a lower unsatiated reward function, and $\widehat{\alpha}(q, r_c)$ is the corresponding robustness function. For any two decision vectors q and q':*

$$\widehat{\alpha}(q, r_c) > \widehat{\alpha}(q', r_c) \quad \textit{implies} \quad \min_{u \in \mathcal{U}[\widehat{\alpha}(q, r_c), \widetilde{u}]} R(q, u) \;>\; \min_{u \in \mathcal{U}[\widehat{\alpha}(q, r_c), \widetilde{u}]} R(q', u) \tag{37}$$

¶ **Meaning of theorems 1 and 2.**

q' = a maximizing decision, such as the EU-maximizing investment $\xi^\star$ in section 2.

r' = a large reward aspiration, for instance the maximum EU $\max_\xi \mathrm{E}_1[U(y_2, \xi^\star)]$.

$r < r'$ is a lower reward aspiration.

q = the robust-satisficing decision at aspiration r, eq.(34), p. 23, for instance $\overline{U} < \max_\xi \mathrm{E}_1(U)$ and $q = \widehat{\xi}(\overline{U})$.

$\widehat{\alpha}(q, r)$ = robustness of q, which is maximal at aspiration r.

All realizations of u up to uncertainty $\widehat{\alpha}(q, r)$ result in reward no less than the aspiration r. That is, $u \in \mathcal{U}[\widehat{\alpha}(q, r), \widetilde{u}]$ implies that $R(q, u) \geq r$.

$\mathcal{U}[\widehat{\alpha}(q, r), \widetilde{u}]$ is the 'domain of robustness' of decision q.

Recall **realized utility difference** $\Delta(y_2)$ in figs. 7 and 8:

- A pessimistic investor may prefer the robust-satisficing over the maximizing strategy because the former performs better when returns are substantially worse than expected.
- Theorems 1 and 2 suggest the same interpretation:
  - The worst reward from the robust-satisficing strategy q, is better than the worst reward from the maximizing strategy q', for all realizations within the domain of robustness of q (theorem 2).
  - The domain of robustness increases as the satisficing aspiration r decreases (theorem 1).

3.3 Opportune Windfalling

¶ We now introduce another **info-gap decision strategy** — opportune windfalling — which provides additional insight into decisions under uncertainty.

¶ The reward function is **upper unsatiated** if:

$$\alpha < \alpha' \quad \text{implies} \quad \max_{u \in \mathcal{U}(\alpha, \widetilde{u})} R(q, u) < \max_{u \in \mathcal{U}(\alpha', \widetilde{u})} R(q, u) \tag{38}$$

Like lower unsatiation, upper unsatiation is a type of monotonicity: the best reward, up to uncertainty α, gets strictly better as the horizon of uncertainty α increases. Upper unsatiation results from the nesting of the sets in the info-gap model $\mathcal{U}(\alpha, \widetilde{u})$.

¶ The **opportuneness** of decision q, with windfalling aspiration that the reward could possibly be as large as r_{w}, is the lowest horizon of uncertainty up to which the aspiration is possible but not necessarily guaranteed:

$$\widehat{\beta}(q, r_{\mathrm{w}}) = \min \left\{ \alpha : \max_{u \in \mathcal{U}(\alpha, \widetilde{u})} R(q, u) \geq r_{\mathrm{w}} \right\} \tag{39}$$

$\widehat{\beta}(q, r_{\mathrm{w}}) = \infty$ by definition if the set in eq.(39) is empty for all values of α. This occurs if and only if $R(q, u) < r_{\mathrm{w}}$ for all values of u at all horizons of uncertainty.

¶ **Preferences of the windfaller:**

$\widehat{\beta}(q, r_{\mathrm{w}})$ is the immunity against windfall, so a small value of $\widehat{\beta}(q, r_{\mathrm{w}})$ is desirable. An **opportune-windfalling** decision at windfalling aspiration r_{w} mimimizes the opportunity function, subject to possible constraints on the choice of q:

$$\widehat{q}_{\mathrm{w}}(r_{\mathrm{w}}) = \arg\min_{q} \widehat{\beta}(q, r_{\mathrm{w}}) \tag{40}$$

3.4 Two More Theorems

¶ Opportuneness trades-off against windfall aspiration:

Theorem 3 *$\mathcal{U}(\alpha, \widetilde{u})$ is an info-gap model, $R(q, u)$ is an upper unsatiated reward function, and $\widehat{\beta}(q, r_{\rm w})$ is the corresponding opportunity function. Let r' be an aspiration whose opportunity is positive with decision vector q: $\widehat{\beta}(q, r') > 0$. Any aspiration for greater windfall is less opportune with q:*

$$r > r' \quad \textit{implies} \quad \widehat{\beta}(q, r) > \widehat{\beta}(q, r') \tag{41}$$

¶ Opportuneness and windfall reward:

Theorem 4 *$\mathcal{U}(\alpha, \widetilde{u})$ is an info-gap model, $R(q, u)$ is an upper unsatiated reward function, and $\widehat{\beta}(q, r_{\rm w})$ is the corresponding opportunity function. For any decision vectors q and q':*

$$\widehat{\beta}(q, r_{\rm w}) < \widehat{\beta}(q', r_{\rm w}) \quad \textit{implies} \quad \max_{u \in \mathcal{U}[\widehat{\beta}(q, r_{\rm w}), \widetilde{u}]} R(q, u) \; > \max_{u \in \mathcal{U}[\widehat{\beta}(q, r_{\rm w}), \widetilde{u}]} R(q', u) \tag{42}$$

¶ Meaning of theorems 3 and 4:

• $q' =$ a maximizing decision, such as the expected-utility-maximizing investment $\xi^\star$ in section 2

• $r' =$ a large reward aspiration, for instance the maximum EU $\max_\xi \mathrm{E}_1[U(y_2, \xi^\star)]$.

• $r > r'$ is a *greater* reward aspiration, a windfall which cannot be guaranteed but which would be a very favorable outcome if it occurred.

• $q =$ the opportune-windfalling decision at this aspiration, eq.(40), p. 26. For instance choose $r > \max_\xi \mathrm{E}_1[U(y_2, \xi^\star)]$ and q as the corresponding opportune-windfalling investment.

$\widehat{\beta}(q, r) =$ the opportuneness of q, the lowest possible immunity to windfall.

• Reward as large as r is possible but not guaranteed if the uncertainty is at least as large as $\widehat{\beta}(q, r)$. That is, there is at least one element $u \in \mathcal{U}[\widehat{\beta}(q, r), \widetilde{u}]$ for which $R(q, u) \geq r$. Smaller uncertainty prevents reward as large as r.

$\mathcal{U}[\widehat{\beta}(q, r), \widetilde{u}]$ ss the 'border of opportunity' of decision q.

• Decision q enables, but does not guarantee, reward as large as r if and only if u is on or beyond the border of opportunity.

• The border of opportunity is different from the domain of robustness, $\mathcal{U}[\widehat{\alpha}(q, r), \widetilde{u}]$, discussed in section 3.2. The domain of robustness is the greatest uncertainty set in which the satisficing aspiration is guaranteed. The border of opportunity is the smallest uncertainty set in which the windfall aspiration is possible. The border of opportunity becomes more distant as $\widehat{\beta}(q, r)$ increases due to nesting of the sets in the info-gap model.

• q, the opportune-windfalling strategy, can attain greater reward than q', the maximizing strategy, in the border of opportunity of q (theorem 4).

• The border of opportunity becomes closer as the windfalling aspiration r decreases (theorem 3).

• Recall **realized utility difference** $\Delta(y_2)$ in figs. 7 and 8:

∘ An optimistic investor may tend to prefer to maximize rather than to robust satisfice because the former performs increasingly better as the returns exceed expectations.

∘ The info-gap opportunity function provides an alternative interpretation, hinging on the concepts of sympathetic and antagonistic immunities. We illustrate this by returning to the example of section 2.

4 Sympathetic Immunities: Two-Period Single-Asset Investment

¶ The info-gap opportunity function, eq.(39), p.26, can be evaluated for the two-period single-asset investment example of section 2. The opportunity function, analogous to the robustness function in eq.(13), p.9, is defined as:

$$\widehat{\beta}(\xi, \overline{U}_{\mathrm{w}}) = \min\left\{\alpha : \max_{y_2 \in \mathcal{Y}(\alpha, \widetilde{y}_2)} U(y_2, \xi) \geq \overline{U}_{\mathrm{w}}\right\} \tag{43}$$

¶ The utility function $u(c)$ increases monotonically in consumption c. Hence the maximum of $U(y_2, \xi)$ in eq.(43), at horizon of uncertainty α with the info-gap model of eq.(12), occurs when y_2 takes its maximal value: $y_2 = (1+\alpha)\widetilde{y}_2$.

¶ Let $\mu_{\mathrm{w}}(\alpha)$ denote the maximum of $U(y_2, \xi)$ in eq.(43):

$$\mu_{\mathrm{w}}(\alpha) = \max_{y_2 \in \mathcal{Y}(\alpha, \widetilde{y}_2)} U(y_2, \xi) \tag{44}$$

$$= u(e_1 - p_1\xi) + \beta u[e_2 + (1+\alpha)\widetilde{y}_2\xi] \tag{45}$$

We are assuming as before that ξ is non-negative.

¶ The opportunity function is obtained by solving $\mu_{\mathrm{w}}(\alpha) = \overline{U}_{\mathrm{w}}$ for α:

$$\widehat{\beta}(\xi, \overline{U}_{\mathrm{w}}) = \begin{cases} 0 & \text{if } \overline{U}_{\mathrm{w}} < U(\widetilde{y}_2, \xi) \\ -1 + \dfrac{1}{\widetilde{y}_2\xi}\left[\left(\dfrac{[\overline{U}_{\mathrm{w}} - u(e_1 - p_1\xi)](1-\gamma)}{\beta}\right)^{1/(1-\gamma)} - e_2\right] & \text{if } \overline{U}_{\mathrm{w}} \geq U(\widetilde{y}_2, \xi) \end{cases} \tag{46}$$

¶ **Compare opportunity, $\widehat{\beta}(\xi, \overline{U}_{\rm w})$, and robustness, $\widehat{\alpha}(\xi, \overline{U}_{\rm s})$, eq.(18), p.11:**

• $\widehat{\alpha}(\xi, \overline{U}_{\rm s})$ is the robustness against failing to achieve at least the satisficing aspiration $\overline{U}_{\rm s}$.

• $\widehat{\beta}(\xi, \overline{U}_{\rm w})$ is also a robustness: the immunity against enabling windfall as large as $\overline{U}_{\rm w}$.

• It is desirable to have:

◦ Large robustness against failure. Thus "bigger is better" for $\widehat{\alpha}(\xi, \overline{U}_{\rm s})$.

◦ Low robustness against windfall. Thus "big is bad" for $\widehat{\beta}(\xi, \overline{U}_{\rm w})$.

• These two immunity functions are **sympathetic** if a change in investment ξ which improves one also improves the other:

$$\frac{\partial \widehat{\alpha}(\xi, \overline{U}_{\rm s})}{\partial \xi} \frac{\partial \widehat{\beta}(\xi, \overline{U}_{\rm w})}{\partial \xi} < 0 \tag{47}$$

The immunity functions are **antagonistic** if they are not sympathetic.

¶ **Compare $\widehat{\beta}(\xi, \overline{U}_{\rm w})$ and $\widehat{\alpha}(\xi, \overline{U}_{\rm s})$:**

• For $\overline{U}_{\rm w} \geq U(\widetilde{y}_2, \xi)$ and $U(0, \xi) \leq \overline{U}_{\rm s} < U(\widetilde{y}_2, \xi)$ one can show that $\widehat{\alpha}$ and $\widehat{\beta}$, eqs.(18) and (46), are related as:

$$\left(\left[1+\widehat{\beta}(\xi, \overline{U}_{\rm w})\right] \widetilde{y}_2 \xi + e_2\right)^{1-\gamma} = \frac{(\overline{U}_{\rm w} - \overline{U}_{\rm s})(1-\gamma)}{\beta} + \left(\left[1-\widehat{\alpha}(\xi, \overline{U}_{\rm s})\right] \widetilde{y}_2 \xi + e_2\right)^{1-\gamma} \quad (48)$$

Differentiating both sides of this equation with respect to ξ yields:

$$1 + \widehat{\beta} + \xi \widehat{\beta}' = A(1 - \widehat{\alpha} - \xi \widehat{\alpha}') \quad (49)$$

where the "prime" implies differentiation and:

$$A = \left(\frac{(1-\widehat{\alpha})\widetilde{y}_2 \xi + e_2}{(1+\widehat{\beta})\widetilde{y}_2 \xi + e_2} \right)^{-\gamma} \quad (50)$$

which is positive.

• From eq.(49): $\widehat{\alpha}(\xi, \overline{U}_{\rm s})$ and $\widehat{\beta}(\xi, \overline{U}_{\rm w})$ do not (ordinarily) have extrema at the same values of ξ, as illustrated in fig. 9, p.32.

• $\widehat{\alpha}(\xi, \overline{U}_{\rm s})$ and $\widehat{\beta}(\xi, \overline{U}_{\rm w})$ will thus tend to be sympathetic (opposing slopes) and antagonistic (agreeing slopes) over various ranges of ξ.

• If the derivative terms in eq.(49) are large, as $\widehat{\alpha}'$ is in fig. 6, p.18, and dominate the other terms, then the extrema are close and the antagonistic range of ξ values will be small. This is a particularly interesting case since the windfalling investor, who chooses the investment with the aim of enabling larger returns than expected, adopts nearly the same investment as the robust satisficing investor.

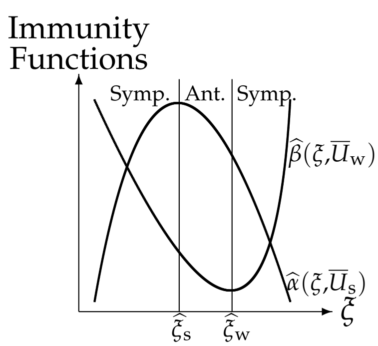

Figure 9: Antagonism and sympathy between robustness and opportunity functions.

5 Equity Premium Puzzle

¶ In section 2 we used a simple investment example to explore the motivation for info-gap robust-satisficing. In this section we use the same example to illustrate that robust-satisficing can provide a promising conceptual framework for understanding the equity premium puzzle.

¶ In subsection 5.1 we briefly summarize a classical statement of the equity premium puzzle.
¶ In subsection 5.2 we derive an info-gap asset pricing relation.
¶ In subsection 5.3 we derive an info-gap analog of the mean-variance frontier and suggest an explanation of the equity puzzle.

5.1 Utility-Maximizing Mean-Variance Frontier

¶ Our discussion in this section is based on Cochrane's clear and succinct exposition of the equity premium puzzle[2]. We consider the 2-period investment problem formulated at the beginning of section 2.

¶ The EU-maximizing investor chooses ξ to maximize the discounted utility, eq.(11). The first order condition on ξ for maximal discounted utility is the basic asset pricing relation:

$$p_1 = \mathrm{E}_1\left[\beta \frac{u'(c_2)}{u'(c_1)} y_2\right] \tag{51}$$

Defining the gross return $R = y_2/p_1$, eq.(51) becomes:

$$1 = \mathrm{E}_1\left[\beta \frac{u'(c_2)}{u'(c_1)} R\right] \tag{52}$$

This leads immediately to the mean-variance frontier which bounds the amount of mean return obtainable at a given level of variance (Cochrane 2001, p.20):

$$\left|\mathrm{E}(R) - R^f\right| \leq \frac{\sigma(m)}{\mathrm{E}(m)} \sigma(R) \tag{53}$$

where $m = \beta u'(c_2)/u'(c_1)$ is the stochastic discount factor and $R^f = 1/\mathrm{E}(m)$ is the risk-free rate of return.

Finally, using the utility function in eq.(17) and assuming that consumption growth is lognormal, the returns on the mean-variance frontier can be approximated as (Cochrane 2001, p.23):

$$\left|\frac{\mathrm{E}(R) - R^f}{\sigma(R)}\right| \approx \gamma\sigma[\ln(c_2/c_1)] \tag{54}$$

where $\sigma[\cdot]$ is the standard deviation of the lognormal consumption growth.

[2] Cochrane, John H., 2001, *Asset Pricing*, Princeton University Press, Princeton, chap. 1.

¶ **Some basic data:**

• The mean and standard deviation of returns to U.S. stocks in the second half of the 20th century were about $\text{E}(R) = 1.09$ and $\sigma(R) = 0.16$.

• The risk-free rate of return in the same period was about $R^f = 1.01$.

• The standard deviation of the log consumption growth $\ln(c_2/c_1)$ was about 1%: $\sigma[\ln(c_2/c_1)] = 0.01$.

¶ Eq.(54) thus implies that $\gamma \approx 50$, which is enormously large risk aversion, in order for the righthand side of eq.(54) to match the Sharpe ratio on the left. As Cochrane writes:

> Clearly, either (1) people are a *lot* more risk averse than we might have thought, (2) the stock returns of the last 50 years were largely good luck rather than an equilibrium compensation for risk, or (3) something is deeply wrong with the model, including the utility function and use of aggregate consumption data. (Cochrane 2001, p.24)

5.2 Info-gap Asset Price Relation

¶ Deriving an info-gap asset-pricing relation:

• The utility function $u(c)$ in eq.(10), p.6, increases monotonically in consumption c. Hence, in the definition of robustness in eq.(13), the minimum discounted utility $U(y_2, \xi)$, at horizon of uncertainty α, occurs when y_2 takes its minimal value: $y_2 = (1-\alpha)\widetilde{y}_2$ (assuming $\alpha \le 1$). Consequently, for any value of ξ, the corresponding robustness is the solution for $\widehat{\alpha}(\xi, \overline{U})$ of:

$$u(e_1 - p_1\xi) + \beta u\left(e_2 + [1 - \widehat{\alpha}(\xi, \overline{U})]\widetilde{y}_2\xi \right) = \overline{U} \tag{55}$$

• Since this relation holds for a continuum of values of ξ, its derivative with respect to ξ is also an equality. In particular, differentiation of eq.(55) at an extreme point of the robustness, where $\partial\widehat{\alpha}(\xi, \overline{U})/\partial\xi = 0$, results in:

$$p_1 = \frac{\beta\widetilde{y}_2 u'(c_2)}{u'(c_1)}[1 - \widehat{\alpha}(\overline{U})] \tag{56}$$

• We have defined $\widehat{\alpha}(\overline{U}) = \widehat{\alpha}(\widehat{\xi}(\overline{U}), \overline{U})$ which is the maximal robustness at utility-aspiration $\overline{U}$. Eq.(56) is the info-gap analog of the classical asset pricing relation, eq.(51), p.35.

¶ Implication of eq.(56):

• Recall the trade-off relation, eq.(20), p.12: robustness decreases as demanded utility increases, as illustrated in fig. 5, p.13.

• That is: high aspiration (large $\overline{U}$) is accompanied by low immunity to uncertainty (small $\widehat{\alpha}(\overline{U})$).

• Furthermore, $\widehat{\alpha}(\overline{U}) = 0$ when $\overline{U}$ is maximal, eq.(21).

• The utility-maximizing investor aspires to attain maximal EU. Hence the robustness curve of fig. 5 contains classical utility maximization as the extreme point at the lower right end of the curve.

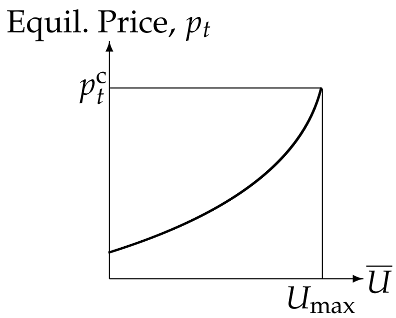

Figure 10: Equilibrium price vs. demanded utility

¶ **Further implications of eq.(56):**

• Eq.(56) is the asset pricing relation when the investor demands discounted utility $\overline{U}$, or equivalently, demands robustness $\widehat{\alpha}(\overline{U})$.

• As the demanded utility $\overline{U}$ gets smaller, the robustness $\widehat{\alpha}(\overline{U})$ gets greater (fig. 5) and the equilibrium price p_t in eq.(56) goes down as illustrated in fig. 10.

• This makes sense: the equilibrium price must be low for an asset from which investors aspire to obtain only a low discounted utility.

• The classical relation is recovered (except for the expectation operator) when $\widehat{\alpha}(\overline{U}) = 0$. This happens when $\overline{U}$ is maximal, which is exactly what the utility-maximizing investor strives to achieve: maximal discounted expected utility. The info-gap pricing curve in fig. 10 contains the classical pricing relation as the extreme point at the upper right end of the curve.

5.3 Info-gap Mean-Variance Frontier

¶ Info-gap analog of the mean-variance frontier:

• Defining the gross return as $R = \widetilde{y}_2 / p_1$, we obtain from eq.(56), p.37:

$$1 = [1 - \widehat{\alpha}(\overline{U})] m R \tag{57}$$

where $m = \beta u'(c_2)/u'(c_1)$ the stochastic discount factor as before.

• How to evaluate mR given severe info-gaps?

• Historical data is available and period-1 moments of the past can be calculated. However, at period 1 the investor has severe Knightian uncertainty about the future returns in period 2, which motivates the info-gap robust-satisficing strategy.

• Replace mR in eq.(57) by its period-1 historical expectation, which we write as:

$$\mathrm{E}_1(mR) = \mathrm{E}_1(m)\mathrm{E}_1(R) + \sigma(m)\sigma(R)\rho_{m,R} \tag{58}$$

where all moments are based on period-1 data.
$\rho_{m,R}$ is the correlation coefficient between m and R and takes values on $[-1, \ 1]$.

• We now obtain the info-gap analog of the mean-variance frontier in which the classical asset price relation, eq.(52), p.35, is replaced by the info-gap asset price relation, eq.(57):

$$\left| \mathrm{E}(R) - \frac{R^f}{1 - \widehat{\alpha}(\overline{U})} \right| \le \frac{\sigma(m)}{\mathrm{E}(m)} \sigma(R) \tag{59}$$

where as before $R^f = 1/\mathrm{E}(m)$.

¶ **Mean-variance frontier with constant risk aversion utility function:**
Let us again adopt the utility function in eq.(17) and assume (based on period-1 data) that the consumption growth is lognormally distributed. Now the upper branch of eq.(59) can be more conveniently written as:

$$\frac{\mathrm{E}(R) - R^f}{\sigma(R)} \approx \frac{\widehat{\alpha}(\overline{U})}{1 - \widehat{\alpha}(\overline{U})} \frac{R^f}{\sigma(R)} + \gamma\sigma[\ln(c_{t+1}/c_t)] \tag{60}$$

¶ **Implications of eq.(60):**

• With the statistical values used earlier we find that a demanded robustness of 7.2%, specifically $\widehat{\alpha}(\overline{U}) = 0.072$, results in a utility exponent of $\gamma = 1$ which is an entirely reasonable level of Arrow-Pratt risk aversion.

• Referring to the info-gap model for uncertainty in the payoff, eq.(12), p.9, we understand that demanding 7% robustness means that the investor requires that any realization of the payoff within ±7% of the estimated value, $\widetilde{y}_{t+1}$, will result in discounted utility no less than $\overline{U}$.

• 7% is a rather modest demand for robustness in stocks. On the other hand it entails satisficing the returns at a level below the maximum.

• The point is to illustrate, with a simple heuristic model, the dramatic effect produced by robust-satisficing the utility and maximizing the robustness, in comparison with the classical approach of maximizing the utility.

• The present simplistic model cannot be expected to provide precise numerical prediction of investor behavior. It does nonetheless indicate the conceptual explanatory power of the info-gap robust-satisficing decision model.

¶ **More implications of eq.(60):**

• The info-gap robustness plays an important role in eq.(60). The righthand side of eq.(60), which contains the info-gap modification to the classical mean-variance relation, expresses large risk aversion without a large value of γ.

• The value of the robustness term, $\widehat{\alpha}/(1-\widehat{\alpha})$, increases as the demanded robustness $\widehat{\alpha}$ increases. In other words, the robust-satisficing decision maker expresses a type of risk aversion which is independent of the utility function.

• Utility-function risk aversion is sometimes understood probabilistically as a preference for a sure average outcome over a bet between extreme outcomes.

• In contrast, info-gap risk aversion focusses on the demand for robustness to uncertainty in the returns. Because of the trade-off between $\widehat{\alpha}(\overline{U})$ and $\overline{U}$, info-gap robustness is obtained only by relinquishing discounted utility, which manifests a distinctly info-gap type of risk aversion.

5.4 Discussion: Knightian Uncertainty, Bounded Rationality and Equity Premia

¶ **Info-gap robust-satisficing** is one possible quantification of two concepts which have long been discussed in economic literature:

- **Knightian uncertainty** and
- **bounded rationality.**
- In addition, the info-gap decision model provides a possible explanation of the **equity premium puzzle.**
- We now discuss these ideas.

¶ Knightian uncertainty.

• Knight's concept: "true uncertainty" for which "there is no objective measure of the probability", as opposed to risk which is probabilistically measurable.[3]

• Similarly, Shackle's "non-distributional uncertainty variable" bears some similarity to info-gap analysis.[4]

• Likewise, Kyburg recognized the possibility of a "decision theory that is based on some non-probabilistic measure of uncertainty."[5]

• Info-gap models provide no opportunity for insurance-like calculations.

• The theory of info-gap uncertainty provides one plausible quantitative model for Knightian uncertainty.

[3]Knight, Frank H., 1921, *Risk, Uncertainty and Profit.* Houghton Mifflin Co. Re-issued by University of Chicago Press, 1971. pp.46, 120, 231–232.

Further discussion of the relation between Knight's conception and info-gap theory is found in Ben-Haim, Yakov, 2001, *Information-gap Decision Theory: Decisions Under Severe Uncertainty,* Academic Press, San Diego, section 12.5).

[4]Shackle, G.L.S., 1972, *Epistemics and Economics: A Critique of Economic Doctrines,* Cambridge University Press, re-issued by Transaction Publishers, New Brunswick, 1992, p.23.

[5]Kyburg, Henry E. jr., 1990, Getting fancy with probability, *Synthese,* 90: 189–203, p.1094.

¶ **Other models of Knightian uncertainty:**

• It is *not* claimed that info-gap models provide the only possible quantification of Knightian uncertainty.

• Gilboa and Schmeidler[6], Epstein and Wang[7], Epstein and Miao[8] and others, achieve uninsurable uncertainty of a clearly Knightian type by replacing a single prior probability distribution with a set of distributions.

• These approaches are Knightian "true uncertainty" since the absence of a probability measure on the set of probability distributions makes the uncertainty uninsurable.

• Nonetheless, an info-gap model of uncertainty is a more extreme departure from probabilistic traditions. In our formulation, preferences are generated by the robustness function without any distribution functions at all.

[6] Gilboa, I. and D. Schmeidler, 1989, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics,* 18: 141–153.

[7] Epstein, Larry G. and Tan Wang, 1994, Intertemporal asset pricing under Knightian uncertainty, *Econometrica,* 62: 283–322.

[8] Epstein, Larry G. and Jianjun Miao, 2003, A two-person dynamic equilibrium under ambiguity, *Journal of Economic Dynamics and Control,* 27: 1253–1288.

¶ **Bounded rationality.**

• **What bounded rationality means:**

◦ The investor's ability to optimize future outcomes is severely limited.

◦ Past asset returns provide only a rough indication of future returns.

◦ The investor's access to information about and understanding of the relevant social and economic forces is far too limited to enable realistic or reliable assessment of maximal future behavior.

• **What bounded rationality implies for the investor:**
The great variability of the returns motivates the choice of a strategy which balances aspiration for utility against aspiration immunity to uncertainty.

• The info-gap robust-satisficing decision strategy does precisely that.

¶ **Info-gap robust-satisficing and economic rationality:**

• Economic rationality: maximize utility (or expected utility).

• Info-gap robust-satisficing violates economic rationality.

• The investor does not attempt to maximize utility, or risk-adjusted utility.

• In the info-gap decision model, utility is satisficed and not maximized.

• What is maximized is robustness to uncertainty.

• The "rationality" of robust-satisficing is expressed in the trade-off of the aspiration for utility against the aspiration for robustness to uncertainty.

• Maximal utility is invariably accompanied by zero robustness. Utility-maximization is entirely unreliable.

¶ **Equity premium.** Kocherlakota concludes his discussion of the equity premium puzzle with the comment:[9]

> The *universality* of the equity premium tells us that, like money, the equity premium must emerge from some primitive and elementary features of asset exchange that are probably best captured through extremely stark models. With this in mind, we cannot hope to find a resolution to the equity premium puzzle by continuing in our current mode of patching the standard models of asset exchange with transactions costs here and risk aversion there. Instead, we must seek to identify what fundamental features of goods and asset markets lead to large risk adjusted price differences between stocks and bonds. (Kocherlakota, 1996, p.67)

¶ **Two implications of info-gap robust-satisficing:**

• **1. Risk aversion:**

◦ Risk aversion is a multi-faceted phenomenon which is not captured entirely by utility-function curvature.

◦ Especially in situations of great Knightian uncertainty, where betting is not a particularly useful concept, risk aversion is expressed in part by the willingness to forego utility in exchange for robustness-to-failure.

[9] Kocherlakota, Narayana R., 1996, The equity premium: It's still a puzzle, *Journal of Economic Literature,* 34: 42–71.

• **2. Rationality:**

◦ The classical axiom of utility maximization can be relaxed without losing touch with economic intuition and data.

◦ The fundamental psychological premise — that utility is desirable — does not imply that maximal utility is most desirable.

◦ We have shown that economic reasoning can be based on, and modelled by, the concept of satisficing the utility and maximizing the robustness with non-probabilistic info-gap models of uncertainty.

◦ The implication of this goes beyond the understanding of equity premia, since all contemporary economic theory is based on the axiom of utility maximization which we have shown to be expendable.