Lecture Notes on

Managing Info-Gap Duration-Uncertainties in Projects

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§ Source material:

• Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd ed., Academic Press, section 3.2.6, chapter 12.
• Yakov Ben-Haim and Alexander Laufer, 1998, Robust reliability of projects with activity-duration uncertainty, *ASCE Journal of Construction Engineering and Management.* 124: 125–132.
• Alexander Laufer and Yakov Ben-Haim, 1998, Robust reliability in project scheduling with time buffering, TME 469.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

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1 Basic Problem

§ A project is characterized by:

- A flow-chart of tasks.
- Uncertainty in the duration of each task. (Alternatively: cost uncertainty.)
- Global requirement: complete project on time (or in budget).

§ Questions:

- How robust is the project to task-duration uncertainty?
- How risky is the project?
- How can the robustness be increased (and the risk reduced)?
  - Re-structuring the project.
  - On-line monitoring.
  - Gathering information.
- How opportune is the project?
  Can windfalls be exploited? How?

2 Project Reliability with a Global Time Buffer: Theory

§ Consider a project whose task flow chart is:

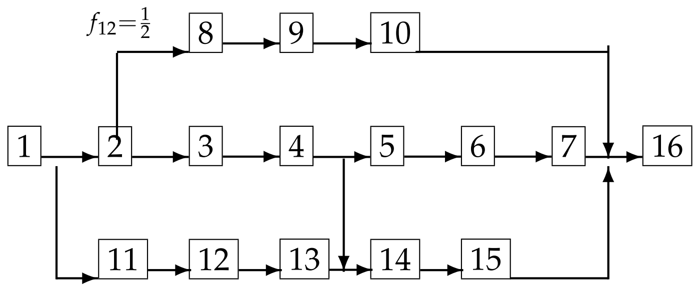

Figure 1: A 16-activity project schedule. Trans. p.blue11

This project has 4 task paths (Trans. p.blue11):
Path 1: $1 \to 2 \to 8 \to 9 \to 10 \to 16$.
Path 2: $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 16$.
Path 3: $1 \to 2 \to 3 \to 4 \to 14 \to 15 \to 16$.
Path 4: $1 \to 11 \to 12 \to 13 \to 14 \to 15 \to 16$.

§ In order to answer the questions in section 1 on page 2 we need:

- Dynamic model: describing the task-path structure and its relation to total project duration.
- Failure criterion.
- Uncertainty model.

§ We first consider the **dynamic model.**
t_n = unknown duration of nth task, $n = 1, \ldots, N$.
$t = (t_1, \ldots, t_N)^T$
There are M paths.

f_{mn} = fractional participation of task n in path m.
m: path.
n: task.
In path m, the task following task n begins when task n is fraction f_{mn} complete.

§ E.g., in path 1 of fig. 1:
task 8 begins when task 2 is 1/2 complete:
$f_{12} = 0.5$.

§ The duration of the mth path, c_m,
equals the sum of the durations of **all tasks**
weighted by their fractional participations in path m:

$$c_m = \sum_{n=1}^{N} f_{mn} t_n, \quad m = 1, \ldots, M \tag{1}$$

For instance, the duration of the 1st path is:

$$c_1 = 1 \cdot t_1 + \frac{1}{2} \cdot t_2 + 1 \cdot t_8 + 1 \cdot t_9 + 1 \cdot t_{10} + 1 \cdot t_{16} \tag{2}$$

Define F = matrix of participation factors $f_{mn} \in \Re^{M \times N}$.
For instance, for fig. 1 (Trans. p.blue12):

$$F = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{3}$$

§ Now the relation between task- and path-durations is:

$$c = Ft \tag{4}$$

The **dynamic model** is the duration of the longest path:

$$T = \|c\| = \max_{1 \le m \le M} |c_m| = \max_{1 \le m \le M} \sum_{n=1}^{N} f_{mn} t_n \tag{5}$$

Note that $\|c\|$ is in fact a vector norm, sometimes called the "zero norm".

§ The **failure criterion:**
the project fails if the duration of the longest path exceeds a critical value:

$$T > t_{\mathrm{c}} \tag{6}$$

§ **Uncertainty model:** weighted fractional variations of task times.

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le w_n h, \quad n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{7}$$

§ This is a family of nested sets.

Two levels of uncertainty:

- At fixed h: t_n, $n = 1, \ldots, N$ are uncertain.

$$\widetilde{t}_n - w_n \widetilde{t}_n h \le t_n \le \widetilde{t}_n + w_n \widetilde{t}_n h \tag{8}$$

- h, the **horizon of uncertainty,** is unknown.

§ The info-gap model in eq.(7) allows negative task durations. Not realistic. **Caution.**

§ Robustness function:

$$
\begin{aligned}
\widehat{h} &= \max h \text{ which precludes failure} && (9)\\
&= \max \{h:\ \text{failure is not possible}\} && (10)\\
&= \max \left\{h:\ T \le t_{\rm c} \text{ for all } t \in \mathcal{U}(h,\widetilde{t})\right\} && (11)\\
&= \max \left\{ h:\ \max_{1\le m\le M} \underbrace{\sum_{n=1}^{N} f_{mn} t_n}_{c_m} \le t_{\rm c} \text{ for all } t \in \mathcal{U}(h,\widetilde{t}) \right\} && (12)\\
&= \max \left\{ h:\ \max_{1\le m\le M} \max_{t\in\mathcal{U}(h,\widetilde{t})} \sum_{n=1}^{N} f_{mn} t_n \le t_{\rm c} \right\} && (13)
\end{aligned}
$$

Recall that, for $t \in \mathcal{U}(h,\widetilde{t})$:

$$
\widetilde{t}_n - w_n \widetilde{t}_n h \le t_n \le \widetilde{t}_n + w_n \widetilde{t}_n h \qquad (14)
$$

Thus:

$$
\begin{aligned}
\max_{t\in\mathcal{U}(h,\widetilde{t})} c_m &= \max_{t\in\mathcal{U}(h,\widetilde{t})} \sum_{n=1}^{N} f_{mn} t_n && (15)\\
&= \sum_{n=1}^{N} f_{mn} \left(\widetilde{t}_n + w_n \widetilde{t}_n h\right) && (16)\\
&= \underbrace{\sum_{n=1}^{N} f_{mn} \widetilde{t}_n}_{\overline{c}_m} + h \underbrace{\sum_{n=1}^{N} f_{mn} w_n \widetilde{t}_n}_{f_m} && (17)\\
&= \overline{c}_m + h f_m && (18)
\end{aligned}
$$

The robustness is obtained by solving for h:

$$\max_{1\le m\le M} (\bar{c}_m + hf_m) = t_{\rm c} \tag{19}$$

We can decompose this according to the separate paths:

$$\widehat{h}_m = \text{robustness of path } m \tag{20}$$

which is the solution for h of:

$$(\bar{c}_m + hf_m) = t_{\rm c} \tag{21}$$

which is:

$$\widehat{h}_m = \frac{t_{\rm c} - \bar{c}_m}{f_m} \tag{22}$$

or zero if this is negative. for each $m = 1, \ldots, M$.
The overall project robustness is the lowest path-robustness:

$$\begin{aligned} \widehat{h} &= \min_{1\le m\le M} \widehat{h}_m && (23) \\ &= \min_{1\le m\le M} \frac{t_{\rm c} - \bar{c}_m}{f_m} && (24) \end{aligned}$$

of zero if this is negative.

3 Calculating Uncertainty Weights

§ Recall the info-gap model of eq.(7):

$$\mathcal{U}(h,\widetilde{t}) = \left\{ t: \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le w_n h, \quad n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{25}$$

We now consider the choice of the uncertainty weights, w_n.

§ There are several obvious ways to go about it. I will mention two.

Notation:
t_i = unknown duration of ith task.
$\widetilde{t}_i$ = estimated duration of ith task.
$\widetilde{t}_{is}$ = estimated shortest duration of ith task.
$\widetilde{t}_{i\ell}$ = estimated longest duration of ith task.
N = number of tasks.

§ **One method** for calculating uncertainty weights generates an **asymmetrical info-gap model.** The info-gap model is:

$$\mathcal{U}(h) \left\{ t: \ \max[0, \ \widetilde{t}_i - (\widetilde{t}_i - \widetilde{t}_{is})h] \le t_i \le \widetilde{t}_i + (\widetilde{t}_{i\ell} - \widetilde{t}_i)h, \ \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{26}$$

Thus t_i belongs to an interval which expands around $\widetilde{t}_i$ as h grows. The interval expands at rate $\widetilde{t}_{i\ell} - \widetilde{t}_i$ above $\widetilde{t}_i$ and at rate $\widetilde{t}_i - \widetilde{t}_{is}$ below $\widetilde{t}_i$. The "max" prevents negative task durations.

§ **Another method** for calculating uncertainty weights generates a **fractional-error info-gap model.** The idea is simply to average the span from shortest to longest estimated duration. The uncertainty weight for the ith task is:

$$w_i = \frac{\widetilde{t}_{i\ell} - \widetilde{t}_{is}}{(1/N)\sum_{j=1}^{N}(t_{j\ell} - t_{js})} \tag{27}$$

Now the info-gap model for duration uncertainty is:

$$\mathcal{U}(h) \left\{ t: \ \left| \frac{t_i - \widetilde{t}_i}{\widetilde{t}_i} \right| \le h w_i, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{28}$$

4 Example: Reliability as a Function of Global Time Buffer

§ Consider the following data for $\widetilde{t}$ and w for the project in fig. 1 on p.3:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\widetilde{t}_n$	1	1	2	3	3	3	2	1	2	3	3	3	1	3	2	1
w_n	1	1	1	1	1	1	1	1	1	1	3	2	2	3	2	1

Table 1: Nominal durations and uncertainty-weights. (Trans. p.blue12)

With this data we can calculate robustnesses as a function of the critical time, t_c:

$\widehat{h}_m =$ path robustnesses.

$\widehat{h} =$ overall project robustness = $\min_m \widehat{h}_m$. See table 2.

t_c	h_1	h_2	h_3	h_4
16	0.88	**0.00**	0.14	0.063
18	1.12	**0.13**	0.24	**0.13**
20	1.35	0.25	0.33	**0.19**

Table 2: Path robustnesses with various allotted activity durations. (Trans. p.blue12)

§ Note the following points:

- At t_c = 16: $\widehat{h}_2 = 0$ because $\widetilde{c}_2 = 16$.
  Thus path 2 is the **nominal-critical path:** Path with shortest estimated duration.
  Based on best estimates, this path would get our greatest attention.
- At t_c = 18: $\widehat{h}_2 = \widehat{h}_4$.
  These two paths have the same robustness at this critical duration.
- At t_c = 20: $\widehat{h}_2 > \widehat{h}_4$. Now:
  the **uncertainty-critical path** (path 4), which determines the overall robustness
  is different from
  the **nominal-critical path** (path 2).

If $t_c = 20$ is acceptable, then path 4, not path 2, should get our greatest attention.

- $\widehat{h}$ increases monotonically, though not linearly, with t_c.
- This reversal of attention between paths 2 and 4 is demonstrated in fig. 2, p.10.

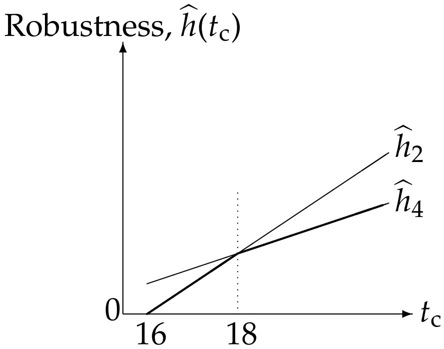

Figure 2: Trade-off of robustness $\widehat{h}_m(t_c)$ against critical time t_c, for two task paths.

5 Example: Real-Time Evaluation of Robustness

§ We continue with the previous example.

We are 2.5 time units after project initiation:

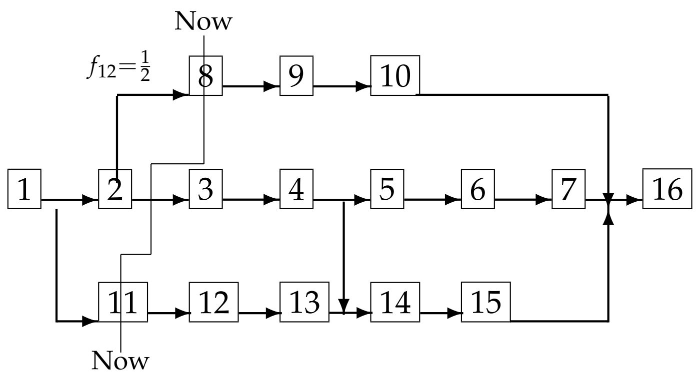

Figure 3: A 16-activity project schedule. The line labeled 'Now' indicates the current status of the project. (Trans. p.blue13)

§ The current situation:

- Task 1 completed after 1.5 time units: 0.5 unit over-run.
- Task 2 completed in 1 time unit as planned.
- Task 8 has been running 0.5 time unit.
- Task 11 has been running 1 time unit.

§ New information in the current situation:

- Task 8 will definitely end in 0.5 time unit.
- Uncertainty in task 11 is reduced somewhat.
- Uncertainty in tasks 5, 6 & 14 is reduced substantially.

This new information is expressed in table 3:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\widetilde{t}_n$	*0*	*0*	2	3	3	3	2	*0.5*	2	3	2	3	1	3	2	1
w_n	*0*	*0*	1	1	*0.5*	*0.5*	1	*0*	1	1	*2*	2	2	*1*	2	1

Table 3: Nominal durations and uncertainty-weights. (Trans. p.blue13)

We now obtain the following path robustnesses (table 4):

t_c Remaining Time	$t_c + 2.5$ Total Time	h_1	h_2	h_3	h_4
14	16.5	1.25	**0.00**	0.23	0.10
15.43	17.93	1.49	**0.13**	0.34	0.17
16.09	18.59	1.60	**0.19**	0.39	0.21

Table 4: Path robustnesses with various allotted activity durations, evaluated during project execution. (Trans. p.blue13)

§ Conclusions from table 4:

- Estimated remaining time: $t_c = 14$ in column 1.
- Total project duration estimated at 16.5 (col. 2); greater than original estimate: 16.
  Due to time overrun of task 1.
- Zero robustness for estimated $t_c = 14$: $\widehat{h}_2(14) = 0$ in column 4.
- Originally, table 2, p.9, and fig. 2, p.10, (see fig. 4 here) reversal of path criticality:
  - ○ $\widehat{h}_2(16) = 0$.
  - ○ $\widehat{h}_2(18) = \widehat{h}_4(18) = 0.13$.
  - ○ $\widehat{h}_4(20) = 0.19 < 0.25 = \widehat{h}_2(20)$.

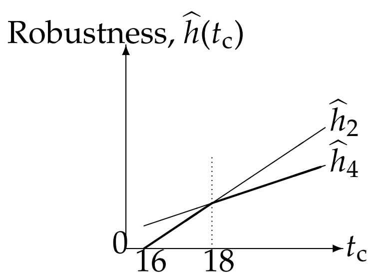

Figure 4: Trade-off of robustness $\widehat{h}_m(t_c)$ against critical time t_c, for two task paths.

- Path 2 is now robust-critical at all estimated t_c's (col. 4): no reversal of path criticality.
- Total durations slightly lower at the same positive robustnesses:
  - ○ $17.93 < 18$ at $\widehat{h} = 0.13$
  - ○ $18.59 < 20$ at $\widehat{h} = 0.19$

6 Enhancing Project Reliability

§ We now consider enhancing project reliability with two types of strategies:

- Reducing uncertainty.
- Re-structuring the project.

6.1 Formulation

§ Consider the following project flow chart:

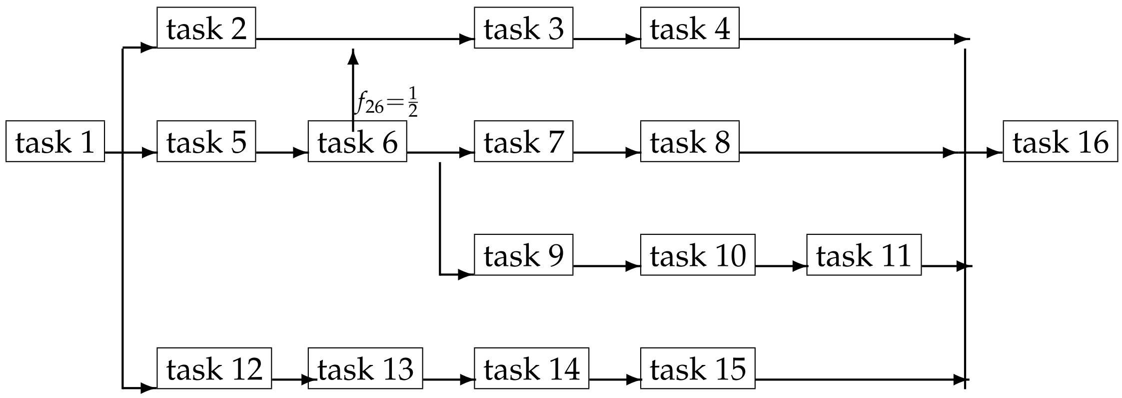

Figure 5: A 16-activity project schedule for section 6. (Trans. p.blue29)

§ The project has 5 task paths (Trans. p.blue29):

Path 1: $1 \to 2 \to 3 \to 4 \to 16$.

Path 2: $1 \to 5 \to 6 \to 3 \to 4 \to 16$.

Path 3: $1 \to 5 \to 6 \to 7 \to 8 \to 16$.

Path 4: $1 \to 5 \to 6 \to 9 \to 10 \to 11 \to 16$.

Path 5: $1 \to 12 \to 13 \to 14 \to 15 \to 16$.

§ Following is the participation matrix (Trans. p.blue29):

$$F = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{29}$$

§ The dynamical model is the duration of the longest path:

$$T = \max_{1 \le m \le M} \sum_{n=1}^{N} f_{mn} t_n \tag{30}$$

§ The failure criterion is:

$$T > t_{\rm c} \tag{31}$$

§ The uncertainty model is:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le w_n h, \quad n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{32}$$

§ Robustness of mth path:

$$\begin{aligned} \widehat{h}_m &= \text{maximum } h \text{ without failure of } m\text{th path} && (33)\\ &= \max \left\{ h : \ \underbrace{\sum_{n=1}^{N} f_{mn} \widetilde{t}_n}_{\widetilde{c}_m} + h \underbrace{\sum_{n=1}^{N} f_{mn} w_n \widetilde{t}_n}_{f_m} \le t_{\rm c} \right\} && (34)\\ &= \max \left\{ h : \ \widetilde{c}_m + h f_m \le t_{\rm c} \right\} && (35) \end{aligned}$$

So:

$$\begin{aligned} \widehat{h}_m &= \text{robustness of path } m && (36)\\ &= \frac{t_{\rm c} - \widetilde{c}_m}{f_m} && (37) \end{aligned}$$

or zero if this is negative. Hence the project robustness is:

$$\widehat{h} = \min_{1 \le m \le M} \widehat{h}_m \tag{38}$$

§ The data for this project are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\widetilde{t}_n$	1	4	6	3	2	3	5	4	4	2	1	2	1	3	1	2
w_n	1	2	2	2	1	2	1	1	0.5	1	1	1	1	1	1	1

Table 5: Nominal durations and uncertainty-weights. (Trans. p.blue30)

The resulting path robustnesses are:

t_c	h_1	h_2	h_3	h_4	h_5
17	0.035	0.058	**0.00**	0.13	0.70
19	**0.10**	0.14	**0.10**	0.25	0.90
21	**0.17**	0.21	0.20	0.38	1.10

Table 6: Path robustnesses with various allotted activity durations. (Trans. p.blue30)

§ Note:

- Path 3 is nominal-critical.
- At $t_c = 19$: $\widehat{h}_1 = \widehat{h}_3$. Other paths more robust.
- At $t_c = 21$: path 1 is uncertainty-critical path. Change of robust-critical path. Fig. 6.
- Large range of robustnesses. E.g., at $t_c = 21$:
  $\widehat{h}_1 = 0.17,\ \widehat{h}_5 = 1.10,\ \dfrac{\widehat{h}_1}{\widehat{h}_5} = 6.5.$
  Meaning: some paths much more reliable than others.

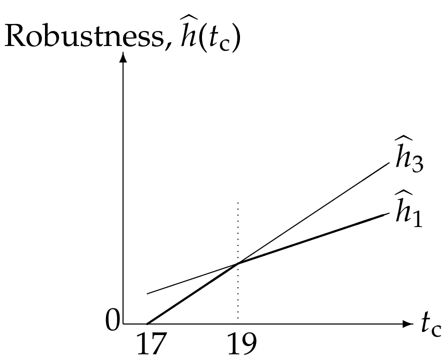

Figure 6: Trade-off of robustness $\widehat{h}_m(t_c)$ against critical time t_c, for two task paths.

6.2 Enhancing Reliability by Reducing Uncertainty

§ Gathering information reduces uncertainty.
We can express this by reducing the uncertainty weights w_n.
Recall how we estimated uncertainty weights earlier: lower and upper time estimates.
Fig. 7 shows all 5 paths vs w_6 (=2 in table 5).

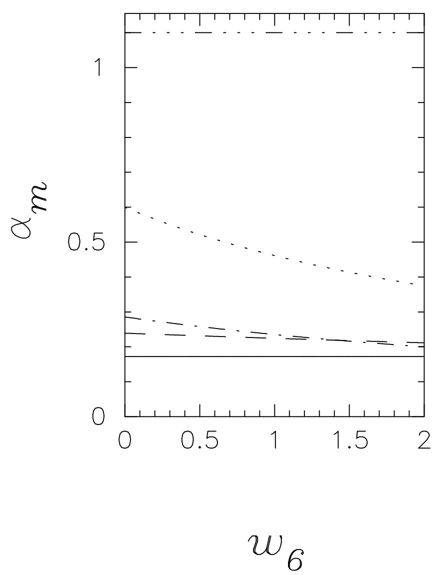

Figure 7: $\widehat{h}_m$ versus w_6. Symbols for paths 1 to 5: (1) solid; (2) dashed; (3) dot-dash; (4) dotted; (5) dash-dot-dot-dot. (Trans. p.blue30)

§ Note:

- Only path-robustnesses $\widehat{h}_2$, $\widehat{h}_3$, $\widehat{h}_4$ vary with w_6.
  Reason: only these paths involve task 6,
  as seen in column 6 in F, eq.(29) on p.14.
- The original critical path, #1, remains critical even at $w_6 = 0$.
- There is no robustness benefit to improved information about task 6.

§ We can influence path 1 by gathering information about task 2,
for which $w_2 = 2$ in table 5 on p.16.
Only path 1 depends on task 2 (See col. 2 of F, eq.(29) on p.14).
Fig. 8 shows $\widehat{h}_1$, $\widehat{h}_2$ and $\widehat{h}_3$ vs w_2.

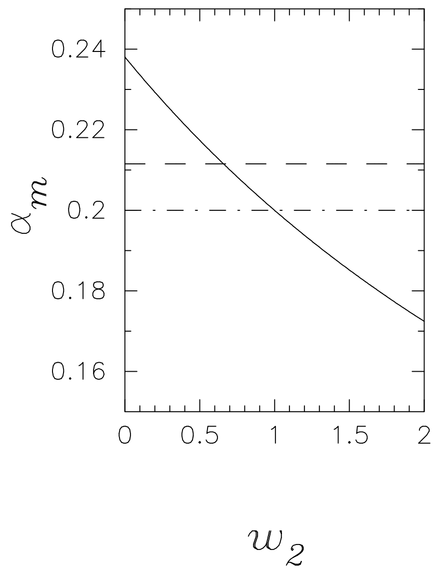

Figure 8: h_m versus w_2. Symbols for paths 1 to 3: (1) solid; (2) dashed; (3) dot-dash. (Trans. p.blue31)

§ Note:

- $\widehat{h}_1$ grows, but not much, as $w_2 \to 0$.
- Path 3 becomes critical for $w_2 \leq 1$.
  Thus not worth reducing $w_2 < 1$.

§ Now gather information about path 3.
Explore effect of reducing w_5, w_6, w_7 and w_8.

§ Suppose we are considering a short-term project,
so that individual task over-runs will be small, about %10.

We ask: How small do these w_n values have to be
in order to achieve the goal of $\widehat{h} > \%10$?
We ask: What project duration is required?

t_c	h_1	h_2	h_3	h_4	h_5
$w_5 = w_6 = w_7 = w_8 = 2$					
17	0.035	0.054	**0.00**	0.11	0.70
19	0.10	0.13	**0.065**	0.22	0.90
21	0.17	0.20	**0.13**	0.33	1.10
$w_5 = w_6 = w_7 = w_8 = 1$					
17	0.035	0.061	**0.00**	0.15	0.70
19	**0.10**	0.14	0.12	0.31	0.90
21	**0.17**	0.22	0.24	0.46	1.10
$w_5 = w_6 = w_7 = w_8 = 0.5$					
17	0.035	0.066	**0.00**	0.19	0.70
19	**0.10**	0.15	0.20	0.38	0.90
21	**0.17**	0.24	0.40	0.57	1.10

Table 7: Path robustnesses with various allotted activity durations. (Trans. p.blue32)

§ Table 7 shows trade-off between:
reducing uncertainty and extending project duration.

§ **1st block:** $w_5 = \cdots = w_8 = 2$:
We achieve $\widehat{h} = 0.13 (\approx 0.10)$ only at $t_c = 21$.
Path 3 is critical.

§ **2nd block:** $w_5 = \cdots = w_8 = 1$:
We achieve $\widehat{h} = 0.10$ at $t_c = 19$.
Path 1 is critical.

§ **3rd block:** $w_5 = \cdots = w_8 = 0.5$:
No further improvement because:
- Path 1 is critical.
- Path 1 is independent of w_5, w_6, w_7 and w_8.

6.3 Enhancing Reliability by Re-structuring

In the original project structure, with $t_c = 21$, path 1 is uncertainty-critical.

Path 1: $1 \to 2 \to 3 \to 4 \to 16$.

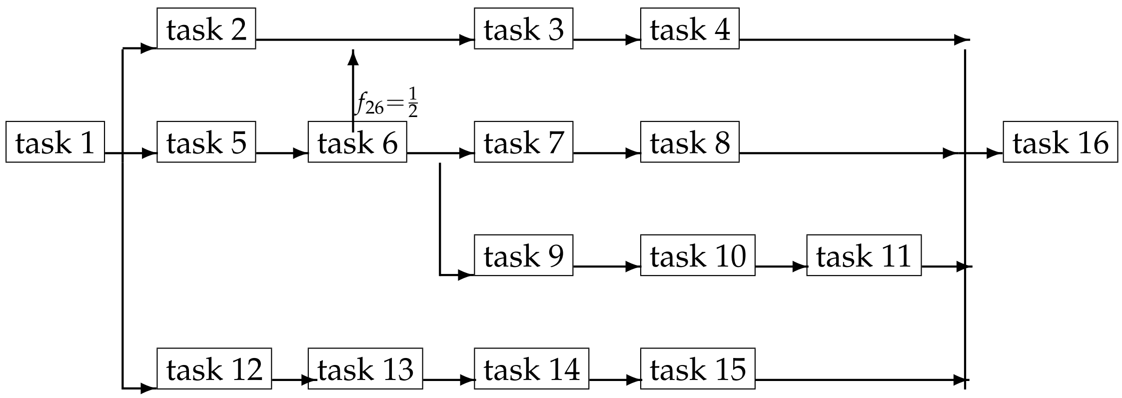

Figure 9: A 16-activity project schedule for section 6. (Trans. p.blue33)

Can we enhance reliability by restructuring this critical path?
Suppose we employ alternative technology to

partially overlap tasks 3 and 4.

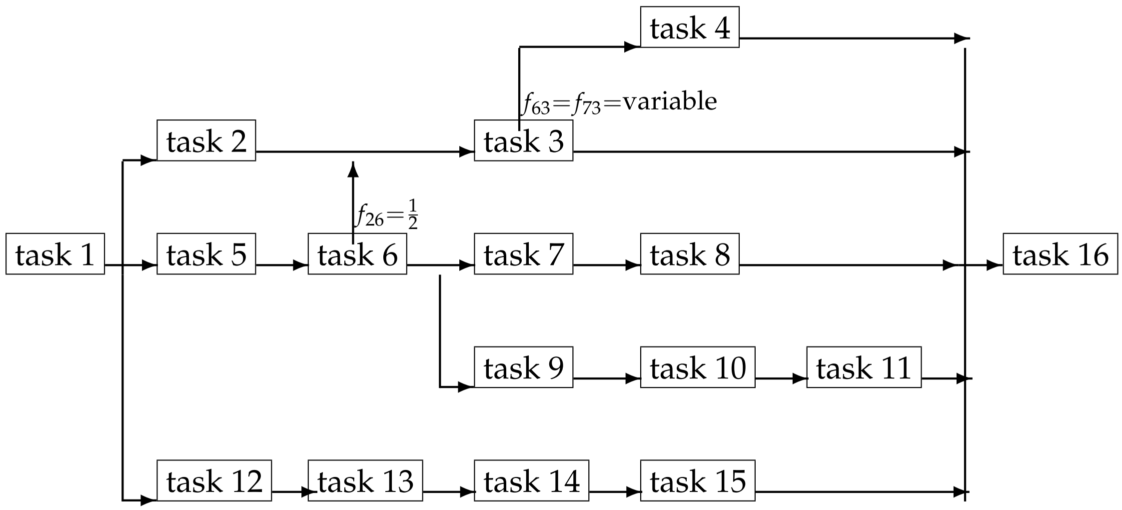

Figure 10: A revised 16-activity project schedule. (Trans. p.blue34)

We now have 7 paths (Trans. p.blue34):
Path 1: $1 \to 2 \to 3 \to 16$.
Path 2: $1 \to 5 \to 6 \to 3 \to 16$.
Path 3: $1 \to 5 \to 6 \to 7 \to 8 \to 16$.
Path 4: $1 \to 5 \to 6 \to 9 \to 10 \to 11 \to 16$.
Path 5: $1 \to 12 \to 13 \to 14 \to 15 \to 16$.
Path 6: $1 \to 2 \to 3 \to 4 \to 16$.
Path 7: $1 \to 5 \to 6 \to 3 \to 4 \to 16$.
The participation matrix is (Trans. p.blue34):

$$F = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & f_{63} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & f_{73} & 1 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

f_{63} = fractional participation of task 3 in path 6.
f_{73} = fractional participation of task 3 in path 7.

$$f_{63} = f_{73} \tag{40}$$

The robustnesses for these 7 paths are in table 8:

t_c	h_1	h_2	h_3	h_4	h_5	h_6	h_7
17	0.17	0.23	**0.00**	0.13	0.70	0.17	0.23
19	0.26	0.33	**0.10**	0.25	0.90	0.26	0.33
21	0.35	0.43	**0.20**	0.38	1.10	0.35	0.43

Table 8: Path robustnesses with various allotted project durations. $f_{63} = f_{73} = 0.5$. $t_c = 21$. (Trans. p.blue35)

§ Note:

- Path 3 is critical at all values of t_c.
- Path 3 was unaffected by the restructuring:
  Path 3: $1 \to 5 \to 6 \to 7 \to 8 \to 16$.
  which is the same as before the structural change.
- The restructuring "robustified" the altered paths,
  and transferred criticality to a previously non-critical path.

§ We now consider the effect on path 6:

Path 6: $1 \to 2 \to 3 \to 4 \to 16$.

and compare with path 3 (critical path for $f_{63} = 0.5$):

Path 3: $1 \to 5 \to 6 \to 7 \to 8 \to 16$.

which is unaffected by the restructuring.

Recall:

$f_{63} = 1 \implies$ no overlap: task 4 starts when task 3 ends.

$f_{63} = 0 \implies$ full overlap: tasks 3 and 4 start together.

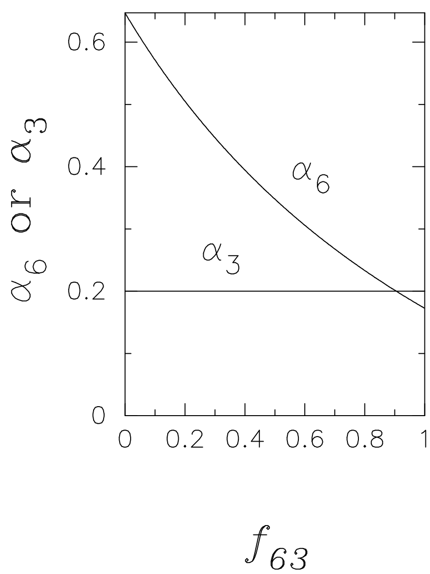

Figure 11: $\widehat{h}_3$ and $\widehat{h}_6(f_{63})$. $t_c = 21$. (Trans. p.blue35)

§ Note:

- $\widehat{h}_6$ increases as overlap increases ($f_{63} : \ 1 \to 0$).
  $\widehat{h}_6(f_{63} = 1) = 0.17$. (No overlap)
  $\widehat{h}_6(f_{63} = 0) = 0.65$. (full overlap)
  Substantial improvement with move from no- to full-overlap.
- $\widehat{h}_3$ is constant since path 3 is unaffected by overlap.
- $\widehat{h}_3 = 0.20$. and $\widehat{h}_3 = \widehat{h}_6$ at $f_{63} = 0.9$
  Hence: no increase in project reliability for overlap $> 10\%$.

§ Now consider that the uncertainty in task 4
may increase with the degree of overlap.
Why? Because task 4 may depend on results obtained in task 3.

§ So let w_4 increase with the degree of overlap:

$$w_4(f_{63} = 1) \;=\; 2$$
$$w_4(f_{63} = 0) \;=\; 5$$

$w_4(f_{63})$ varies linearly with f_{63}.

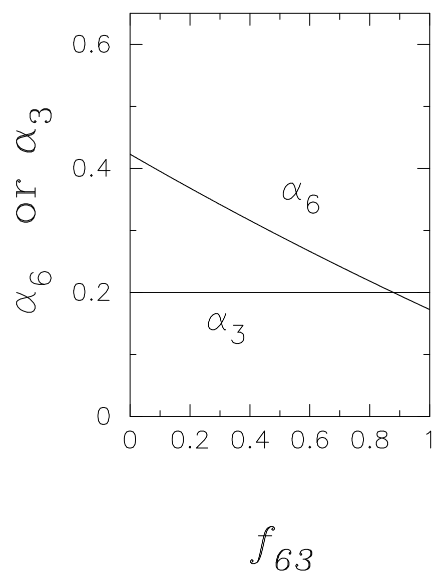

Figure 12: h_3 and $h_6(f_{63}, w_4)$. $t_c = 21$. (Trans. p.blue36)

§ Note:

- $\widehat{h}_6(f_{63} = 0) \;=\; 0.41$ as opposed to $\widehat{h}_6(f_{63} = 0) \;=\; 0.65$ in fig. 11 on p.24.
  So improvement is still good, but not as good.
- $\widehat{h}_3 = \widehat{h}_6(f_{63})$ at very nearly the same f_{63} (~ 0.9).
  So virtually no impact on the transfer of criticality to path 3.
  Still, greatest useful overlap is $\sim$10%.

7 Enhancing Reliability with Local Time Buffers

§ We now consider a multi-task project as before,
but now we are concerned with
local stability.
That is, we consider failure as:
time over-run of any individual task.
Of course, we are still concerned with over-all project duration.

§ The basic idea is to allocate **local time buffers** to each task.

§ Define:
t_c = duration for completion of project.
$\widetilde{c}_m$ = nominal duration of path m.
Hence:
$t_c - \widetilde{c}_m$ = amount of "buffer time" which can be allotted among the tasks of path m.
The question: how to distributed this buffer among the tasks?
We will formulate the basic ouline of this problem,
but we will not study its detailed solution.

§ There are N tasks, for which:

$$t_n = \text{unknown \textbf{actual} duration of task } n \tag{41}$$

$$t = (t_1, \ldots, t_N)^T \tag{42}$$

$$\widetilde{t}_n = \text{known \textbf{nominal} duration of task } n \tag{43}$$

$$\widetilde{t} = (\widetilde{t}_1, \ldots, \widetilde{t}_N)^T \tag{44}$$

§ The uncertainty model is, as before:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \leq w_n h, \quad n = 1, \ldots, N \right\}, \quad h \geq 0 \tag{45}$$

§ Let b_n = **buffer time** following task n.
That is, b_n is the amount of spare time
during which we plan to be idle,
following completion of task n.
No delay results if task n completes during b_n.
Define:

$$b = (b_1, \ldots, b_N)^T \tag{46}$$

§ The time over-run of task n is:

$$\delta_n(t_n) = \max \left\{ t_n, \widetilde{t}_n + b_n \right\} - (\widetilde{t}_n + b_n) \tag{47}$$

§ As before, we need 3 components for reliability analysis:

- Dynamic model of the system.
- Failure criterion.
- Uncertainty model: eq.(45) on p.27.

§ **Failure:** If any single task exceeds its allotted time $\widetilde{t}_n + b_n$
by more than a specified amount $\Delta_{c,n}$.
That is, failure occurs if:

$$\max_{1 \le n \le N} [\delta_n(t_n) - \Delta_{c,n}] \ > \ 0 \tag{48}$$

$\Delta_{c,n}$ can be chosen as any non-negative value.
$\Delta_{c,n}$ can be different for different tasks.

§ **Dynamic model:**
The failure criterion is applied "locally", at each task.
Hence the path structure does not directly affect success or failure.
The dynamic model is simply the vector t of task durations.

§ **Robustness** of task n is the greatest tolerable value of h:

$$\widehat{h}_n = \max \left\{ h : \max_{t_n \in \mathcal{U}(h, \widetilde{t})} \delta_n(t_n) \le \Delta_{c,n} \right\} \tag{49}$$

This is obtained by solving the following relation for h:

$$\max_{t_n \in \mathcal{U}(h, \widetilde{t})} \delta_n(t_n) = \Delta_{c,n} \tag{50}$$

§ Max over-run of task n, up to uncertainty h:

$$\max_{t_n \in \mathcal{U}(h, \widetilde{t})} \delta_n(t_n) = \max \left\{ (1 + w_n h) \widetilde{t}_n, \ \widetilde{t}_n + b_n \right\} - (\widetilde{t}_n + b_n) \tag{51}$$

where we understand that:
$(1 + w_n h)\widetilde{t}_n$ = greatest duration of task n possible at horizon of uncertainty h, e.g. allowed by $\mathcal{U}(h, \widetilde{t})$.
$\widetilde{t}_n + b_n$ = greatest nominal duration of task n.

Hence the robustness of task n is:

$$\widehat{h}_n = \frac{b_n + \Delta_{c,n}}{\widetilde{t}_n w_n} \tag{52}$$

The overall robustness of the project is:

$$\widehat{h} = \min_{1\le n\le N} \widehat{h}_n \tag{53}$$

$$= \min_{1\le n\le N} \frac{b_n + \Delta_{c,n}}{\widetilde{t}_n w_n} \tag{54}$$

§ We would like to choose the buffer times b to maximize $\widehat{h}$.

One approach is to use a 'Robin Hood' principle:

- Take buffer time away from very robust tasks.
- Give buffer time to very vulnerable tasks.
- Continue this until the robustnesses of the tasks are as equal as possible.

We will not pursue this optimization problem.