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Approval and Plurality Voting with Uncertainty: Info-Gap Analysis of Robustness

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Abstract Voting algorithms are used to choose candidates by an electorate. However, voter participation is variable and uncertain, and projections from polls or past elections are uncertain because voter preferences may change. Furthermore, electoral victory margins are often slim. Variable voter participation or preferences, and slim margins of decision, have implications for choosing a voting algorithm. We focus on approval voting (AV) and compare it to plurality voting (PV), regarding their robustness to uncertainty in voting outcomes. We ask: by how much can voting outcomes change without altering the election outcomes? We see fairly consistent empirical differences between AV and PV. In single-winner elections, PV tends to be more robust to vote uncertainty than AV in races with large victory margins, while AV tends to be more robust at low victory margins. Two conflicting concepts — approval flattening and approval magnification — explain this tendency for reversal of robust dominance between PV and AV. We also examine the robustness to vote uncertainty of PV in elections for proportional representation of parties.

1 Introduction

Voting algorithms in democracies are used to choose candidates, legislators, and executive officers by an electorate. However, the extent of participation in elections by voters may vary widely and can be highly uncertain. Projections from polls or past elections to future elections are uncertain due to the potential for voters to change their preferences. Furthermore, the margins by which electoral decisions are made are often slim. Variable voter participation or preferences, together with slim margins of decision, have important implications for the choice of an electoral voting algorithm. We focus on approval voting and compare it to plurality voting, with respect to their robustness to uncertainty in voting outcomes. The central question we address is: by how much can voting outcomes change without altering the election outcomes? We will see important and fairly consistent empirical differences between approval voting and plurality voting.

Voter turnout varies greatly between countries, from less than 50% in Switzerland and other countries, to over 90% in Australia and elsewhere (Stockemer, 2017). However, the variability in turnout for parliamentary elections also sometimes varies substantially over time in the same country, displaying both trends and fluctuations. The percentage of the voter-age population (VAP) that turned out for parliamentary elections in France changed from 71% to 61% and back to 71% in 1958, 1962 and 1967, respectively. The extent of VAP participation of the Israeli electorate in elections of Knesset (parliament) members reached a high of 86.9% (in 1949), drifted from 82.8% (in 1955) to 78.8% (in 1999), reached a low of 63.5% (in 2006), and rose to 72.3% in 2015 (Knesset, 2019; Atmor and Friedberg, 2015). VAP turnouts in The Netherlands were 88%, 92% and 78% in 1963, 1967, and 1971. VAP turnouts in the United Kingdom were 75%, 69% and 58% in 1992, 1997 and 2001. VAP turnouts in Japan were 66%, 45% and 60% in 1993, 1995 and 1996. VAP turnouts in Taiwan were 57%, 74%, 66% and 74% in 2008, 2012, 2016 and 2020. More examples abound around the world, especially in countries subject to more chaotic conditions (Institute for Democracy and Electoral Assistance). Turnout in elections for the European Parliament also display wide variability between countries (Blondel, Sinnott and Svensson, 1997, 243)

Voter attitudes can have a major impact on voter turnout, and in surveys conducted in relation to elections for the European Parliament, "the main reason given for abstention is lack of interest; . . . .

Three other reasons are fairly frequently given: political distrust or dissatisfaction, lack of knowledge and dissatisfaction with the electoral system." (Blondel, Sinnott and Svensson, 1997, 265–266). In a game-theoretic analysis, Palfrey and Rosenthal (1985, 62) concluded that the extent of voter participation can be strongly influenced by the degree of uncertainty about preferences and costs. Voters are also motivated to vote, or not, depending on a sense of civic duty or ethical obligation (Feddersen and Sandroni, 2006, 1271). Voter participation is influenced by spending by candidates, voters' socio-economic status, and the electoral margin by which an outcome is obtained (Hogan, 1999, 403, 405; Mueller and Stratmann, 2003, 2129; Silberman and Durden, 1975, 107). However, the evidence for influence of electoral margin on voter participation in Canadian Federal elections is "mixed" according to Endersby, Galatas and Rackaway (2002, 610). Levine and Palfrey (2007, 143) report that voter turnout goes down in large electorates, and goes up among voters supporting less popular alternatives. These numerous factors are all inter-related and uncertain, especially perceptions of status, knowledge and interest, and anticipations of outcomes.

Popular vote margins are often slim. The margin of victory in popular votes in many U.S. Presidential elections is often small: 0.17% for Kennedy in 1960, 0.70% for Nixon in 1968, −0.51% for George W. Bush in 2000 (Bush won in the electoral college by 0.37%).

Diverse algorithms are available for popular elections. Plurality or majority are the most common, but other algorithms have been proposed.

In 1770 Jean-Charles de Borda "proposed 'election by order of merit,' now known as Borda's rule. Under this method, each voter ranks the candidates in order, and each candidate is awarded a number of votes (from that voter) equal to the number of other candidates ranked below him; the candidate receiving the greatest total number of votes wins the election." (Weber, 1995, 39).

In the 1970s Weber proposed an election algorithm called approval voting (AV) whereby "each voter is allowed to cast a single vote for each of as many candidates as he or she wishes—that is, the voter votes for all candidates of whom the voter 'approves.' The candidate receiving the greatest total number of votes is declared the winner." (*ibid*, 40). A modification of approval voting limits the number of approvals that each voter can allot. Ordinary plurality voting results when this approval limit equals one. Brams and Fishburn (1978) and Brams (2008) discuss advantages of AV, and Niemi (1984) raises some substantial objections.

Tabarrok (2001, p.284) notes that pre-election polls may differ from final election results because "some voters changed their minds by the time of the election" and considers the robustness of results to plausible changes in outcomes. We will study in depth the robustness of approval voting to uncertainty in vote outcomes.

Dehez and Ginsburgh (2019, p.13) write that "Approval voting has its advantages and drawbacks like any other preference aggregation method, although most of its advantages cannot be formalized." They note that a challenge in assessing AV is that it depends on the strength of voters' preferences which are uncertain. This uncertainty in voter preferences is related to uncertainty in voting outcomes.

In short, electoral voting entails many deep uncertainties. This motivates our focus in this paper on how voting algorithms perform under uncertainty. We focus specifically on approval voting (AV) and compare it with conventional plurality voting (PV) in single-winner elections. In section 2 we formulate the info-gap representation of uncertainty in AV outcomes. In section 3 we develop the robustness to uncertain AV outcomes with a single winner, and in section 4 we do the same for PV. We discuss an empirical example in section 5. In section 6 we briefly examine the robustness of PV in an election whose outcome determines the proportional representations of a number of parties.

Section 7 concludes.

2 Info-Gap Representation of Uncertainty in Approval Votes

Our analysis is based on a non-probabilistic representation of uncertainty, and the concept of robustness, as developed in info-gap decision theory. An intuitive and non-technical discussion of info-gap theory appears in Ben-Haim (2018); mathematical development of the methodology is presented in Ben-Haim (2006, 2010). Many citations appear on the website info-gap.com.

As explained in section 1, approval voting is an election algorithm in which each voter may cast a single vote for each of as many candidates as the voter wishes. The winner is the candidate who received the greatest number of such "approvals". In this section we develop an info-gap representation for the uncertainty in these approval votes.

The number of candidates is N, which is greater than 2, and the total number of participating voters is T. The results of a specific AV election yielded $\widetilde{a}_n$ approvals for candidate n, for $n = 1, \ldots, N$, which we denote with the vector $\widetilde{a}$. The candidate with the greatest number of approvals won. However, we want to know if this result is robust to uncertainty in the voting. We now formulate the non-probabilistic info-gap representation of uncertainty. In the next section we formulate the robustness to uncertainty.

As explained in section 1, many factors impact voter participation and preferences. As a consequence, the AV results that are actually observed could have been quite different, so we face deep uncertainty about the vote outcome. That is, the observed AV results, denoted by the vector $\widetilde{a} = (\widetilde{a}_1, \ldots, \widetilde{a}_N)$, could have been different as denoted by a vector $a = (a_1, \ldots, a_N)$. We have no reliable probabilistic information about this uncertainty of the observed outcome, so we now develop a non-probabilistic info-gap representation of this uncertainty.

We suppose that, for each outcome $\widetilde{a}_i$, we have a rough estimate of its potential variation represented by a positive uncertainty weight w_i. Thus the approval votes for candidate i could vary by $\pm w_i$ or more, but cannot be negative or greater than the number of participating voters (we treat elements of a as non-negative *real* numbers for simplicity, though in fact they must be integers). We represent this uncertainty with the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ a : \ 0 \le a_n \le T, \ \left| \frac{a_n - \widetilde{a}_n}{w_n} \right| \le h, \ n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{1}$$

$\mathcal{U}(h)$ is the set of all allowed values of AV outcome vectors, a, that deviate fractionally from the observed vector, $\widetilde{a}$, by no more than h. The set becomes more inclusive as h grows, so h is called the horizon of uncertainty. The info-gap model of uncertainty is not a single set, but rather the family of nested sets $\mathcal{U}(h)$ for $h \ge 0$. The info-gap model represents the uncertainty non-probabilistically due to the limited available information. Furthermore, there is no known worst case.

There are many types of info-gap models of uncertainty, of which eq.(1) is the most relevant for the present situation, but they all obey the contraction and nesting axioms (Ben-Haim, 2006, 2010).

The *contraction axiom* asserts that, in the absence of uncertainty, the votes take the observed values $\widetilde{a}$. Formally:

$$\mathcal{U}(0) = \{\widetilde{a}\} \tag{2}$$

The *nesting axiom* asserts that the sets become more inclusive as the parameter h increases:

$$h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h') \tag{3}$$

3 Robustness of Approval Voting with a Single Winner

The condition for candidate i to beat candidate j by a non-negative margin δ is:

$$a_i - a_j > \delta \tag{4}$$

The significance of the victory margin, δ, will become evident after we derive the robustness function.

Regarding the competition between candidates i and j, the robustness to uncertainty in the approval votes is the greatest horizon of uncertainty, h, up to which all voting outcomes a in the uncertainty set $\mathcal{U}(h)$ of eq.(1) satisfy the requirement in eq.(4):

$$\widehat{h}_{ij}(\delta) = \max\left\{h: \left(\min_{a\in\mathcal{U}(h)} \left(a_i - a_j\right)\right) > \delta\right\} \tag{5}$$

We are considering situations in which one candidate beats the other. Let i denote the putatively winning candidate, so $\widetilde{a}_i > \widetilde{a}_j$.

The following expression for the robustness function is derived in appendix A:

$$\widehat{h}_{ij}(\delta) = \frac{\widetilde{a}_i - \widetilde{a}_j - \delta}{w_i + w_j} \tag{6}$$

or zero if this is negative, where $\delta \geq 0$.

Eq.(6) asserts that the robustness to uncertainty increases as the observed margin of victory, $\widetilde{a}_i - \widetilde{a}_j$, increases, and decreases as the total uncertainty weight, $w_i + w_j$, increases.

Like all info-gap robustness functions, $\widehat{h}_{ij}(\delta)$ in eq.(6) displays the trade off and zeroing properties, and which will be seen in the examples in sections 5 and 6. The original question was: how much can the AV results change without altering the outcome? The zeroing property asserts that we have no robustness to uncertainty if we require the observed margin of victory. This means that the observed margin of victory is not a reliable measure of voter preference for the winner, due to deep uncertainty in voting outcomes.

The trade off property asserts that only lower margins of victory have positive robustness to uncertainty in voting outcomes. That is, the robustness, $\widehat{h}_{ij}(\delta)$, increases (which is desirable) as the required victory margin, δ, decreases (which is undesirable). The robustness function thus supports the choice of a reliable margin of victory: a value of δ for which the corresponding robustness is large. Note, however, that the robustness in eq.(6) cannot exceed the ratio of the observed margin of victory, $\widetilde{a}_i - \widetilde{a}_j$, to the sum of the uncertainty weights, $w_i + w_j$. Thus the maximal robustness will be much less than one in a close race with large uncertainty weights. This is especially acute when the uncertainty weights, w_i and w_j, are estimated as the vote counts themselves, $\widetilde{a}_i$ and $\widetilde{a}_j$. In other words, large robustness and high confidence in outcomes will not be possible in all situations.

We are considering approval voting in multi-candidate elections with a single winner. However, while only a single candidate actually wins the election, other races within the multi-candidate election may be significant. For instance, a single candidate will win a multi-candidate primary election, but losing candidates may still run in later primary elections or may claim positions or privileges in subsequent political developments. The confidence one has that margins of victory among these losing candidates will recur is relevant to these circumstances. For instance, a losing candidate whose robustness of loss is large may withdraw, but not if the robustness is small. In the latter case — small robustness for loss — the loser may reason that small changes in voter behavior in the next round could yield victory.

4 Robustness of Plurality Voting with a Single Winner

We now formulate the robustness function for plurality voting, and in section 5 we compare it with approval voting with various real election results.

The number of candidates is N and the total number of voters is T, where each voter can vote for no more than 1 candidate. In analogy to eq.(1), the info-gap model for uncertainty in the voting outcomes is:

$$\mathcal{U}(h) = \left\{ a : \ 0 \le a_n, \ \sum_{n=1}^{N} a_n \le T, \ \left| \frac{a_n - \widetilde{a}_n}{w_n} \right| \le h, \ n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{7}$$

Consider a race where candidate i beats candidate j in the official outcome, that is $\widetilde{a}_i > \widetilde{a}_j$. The definition of the robustness of this outcome is eq.(5) where we now use the info-gap model in eq.(7). In appendix B we derive the robustness function for the special case that the uncertainty weights are proportional to the observed votes:

$$w = \gamma \widetilde{a} \tag{8}$$

for some $\gamma > 0$. The result is:

$$\widehat{h}_{ij}(\delta) = \frac{\widetilde{a}_i - \widetilde{a}_j - \delta}{(\widetilde{a}_i + \widetilde{a}_j)\gamma} \tag{9}$$

or zero if this is negative. Note that eq.(8) implies that this is algebraically the same as eq.(6), the robustness for approval voting, though different constraints apply to the voting outcomes.

5 Comparing Approval and Plurality Voting with a Single Winner

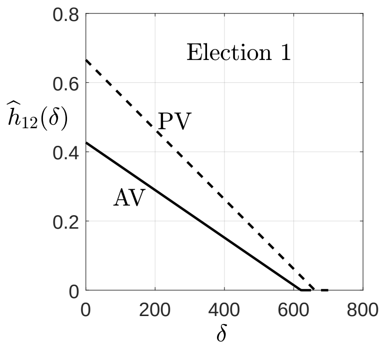
In this section we compare the robustness to vote uncertainty of approval and plurality voting, by analyzing actual outcomes in several different single-winner elections. We adopt the condition in eq.(8) for the uncertainty weights of both AV and PV, so the approval and plurality robustness functions, eqs.(6) and (9), are algebraically the same though they apply to different outcome vectors. We choose $\gamma = 1$ so the uncertainty weights are the voting outcomes. We will see that PV tends to be more robust to vote uncertainty than AV in races with large margins of victory, while AV tends to be more robust at low margins of victory. We will suggest two competing concepts — approval flattening and approval magnification — to explain this tendency for a reversal of robust dominance between PV and AV.

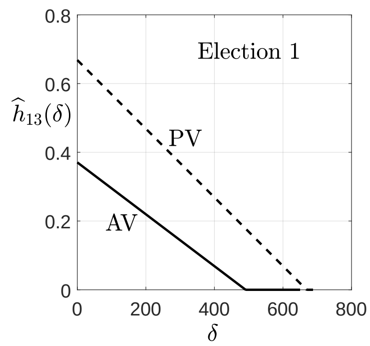
5.1 A First Look at Some Evidence

Fishburn and Little (1988) discuss results of 3 elections in The Institute of Management Sciences (TIMS) in which participants returned both plurality and approval voting ballots. Results of 2 of these elections are reproduced here in table 1.

Candidate	First Election		Second Election	
	Plurality	Approval	Plurality	Approval
A	166	417	386	558
B	827	1038	551	713
C	835	908	599	768

Table 1: Results of two TIMS elections, from Fishburn and Little (1988).





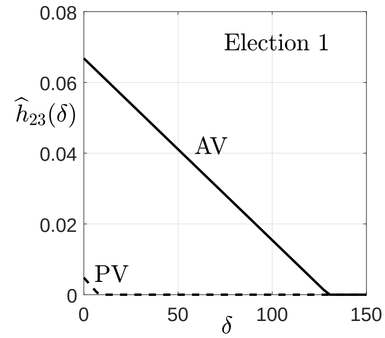
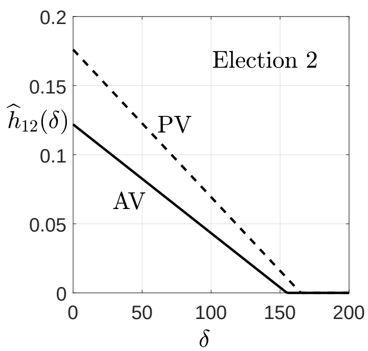
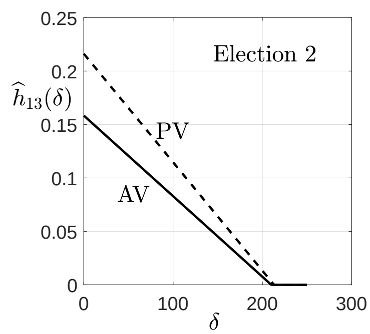


Figure 1: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates A and B in election 1.

Figure 2: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates A and C in election 1.

Figure 3: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates B and C in election 1.





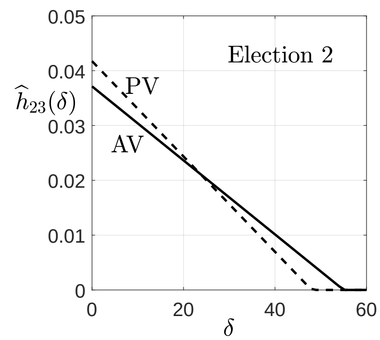


Figure 4: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates A and B in election 2.

Figure 5: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates A and C in election 2.

Figure 6: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates B and C in election 2.

Figs. 1–3 show robustness curves for the first TIMS election, based on eqs.(6), (8) and (9) with $\gamma = 1$. Note that the vertical and horizontal scales are not the same in all figures, due to the widely different range of values. Two features are particularly prominent. First, plurality voting (PV) is more robust than approval voting (AV) in the races of candidates A against B, and A against C, but not for B against C. Second, the B-C race has much smaller margin of victory and hence much lower robustness than the other two races. Recall also that the PV and AV outcomes are different in the B-C race of election 1: B wins according to AV, while C wins according to PV. The other outcomes of election 1 are the same according to AV and PV.

A similar picture emerges from figs. 4–6 for election 2 (note again the different scales in the figures). PV is more robust than AV in the large-margin races: A vs B and A vs C. In contrast, and like fig. 3, PV and AV both have low robustness in the B vs C race where the margin of victory is low. In fact, in the B vs C race the robustness curves are close and even cross one another. AV and PV agree on all three outcomes in election 2.

What do the robustness numbers mean? This will lead us to the ideas of zeroing and trade off. The **zeroing property** asserts that observed voting outcomes have no robustness against uncertainty.

The **trade off property** is that the robustness, $\widehat{h}_{ij}(\delta)$, goes down (gets worse) as the critical value, δ, increases (gets more demanding). This is demonstrated by the negative slopes of the robustness curves.

Take, for example, the B vs C race in the first TIMS election, fig. 3. From table 1 we see that candidate C beat B by 8 votes in the PV election, but B beat C by 130 votes in the AV election. The robustness to voter uncertainty is precisely zero for both of these outcomes (this is the zeroing property). Only lower margins of victory have positive robustness, as expressed by the negative slopes of the robustness curves (the trade off property). For instance, looking again at fig. 3, C beats B by 5 votes with robustness of 0.064 in the AV election, but B beats C by 5 votes with robustness of only 0.0018 in the PV election. This means that the observed outcomes could vary by $\pm 6.4\%$ in an AV election and C would still beat B by no less than 5 votes. In contrast, PV outcomes could vary by only $\pm 0.18\%$ in a PV election and B would still beat C by no less than 5 votes. While robustness to 6.4% variation in voting outcomes is not large, it is greater than 0.18%. This means that we would have more confidence in the AV outcome (victory for C) than in the PV outcome (victory for B), though neither outcome is highly robust to voting uncertainty.

We now examine these results more closely by introducing two concepts: approval flattening in section 5.2, and approval magnification in section 5.3.

5.2 Robust Dominance of PV at Large Margins of Victory: Approval Flattening

The robust dominance of plurality voting over approval voting, when the margin of victory is substantial, suggests the phenomenon of 'approval flattening', by which we mean that AV outcomes are less decisive in two senses.

First, approval flattening means that the AV margin of victory decreases compared to PV. Brams and Nagel (1991, p.14) wrote that "Generally speaking, AV will result in relatively higher vote totals for 'underdog' candidates, [because] … under AV, voters need not worry about 'wasting' their votes if they want to support someone thought to have little or no chance of winning." Vote totals are larger, and victory margins may tend to be smaller, in AV elections.

The second aspect of approval flattening is that the ratio of the margin of victory to the total vote count decreases compared to PV. These two aspects of approval flattening are reflected in reduced robustness of approval voting to vote-count uncertainty, as will be explained shortly.

Approval flattening does not necessarily occur, in particular due to the phenomenon of approval magnification to be discussed in section 5.3. However, one might expect that approval flattening will tend to occur because voters can express approval for additional candidates in approval voting, for whom they cannot express approval in plurality voting.

Approval flattening is manifested in two aspects of the robustness curves in figs. 1, 2, 4 and 5. From eqs.(6) and (9) we see that the horizontal intercept of each robustness curve is the observed margin of victory, $\widetilde{a}_i - \widetilde{a}_j$. This margin is smaller in each of these four AV races than in the corresponding PV races. The vertical intercept of each robustness curve is $(\widetilde{a}_i - \widetilde{a}_j)/(\widetilde{a}_i + \widetilde{a}_j)$. This ratio is also lower for each of the four AV races. Approval flattening thus shifts the AV robustness curve to the left and downward with respect to the PV robustness curve, which is precisely what we see in figs. 1, 2, 4 and 5.

While approval voting has normative attributes in its favor, as explained by Brams and others, it often displays lower robustness to uncertainty in voter turnout due to approval flattening, as seen in figs. 1, 2, 4 and 5. However, we have yet to explain the contrary results of figs. 3 and 6, which depend on the phenomenon of approval magnification.

5.3 Robust Dominance of AV at Low Margins of Victory: Approval Magnification

Fishburn and Little (1988) consider a third TIMS election between 5 candidates in which the plurality votes determine two positions to be filled. The PV and AV results are reproduced in table 2.

Candidate	Third Election	
	Plurality	Approval
A	679	669
B	937	956
C	651	715
D	670	685
E	391	483

Table 2: Results of the third TIMS election, from Fishburn and Little (1988).

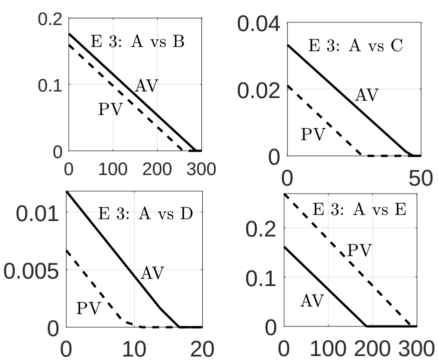


Figure 7: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates A vs B, C, D and E in election 3.

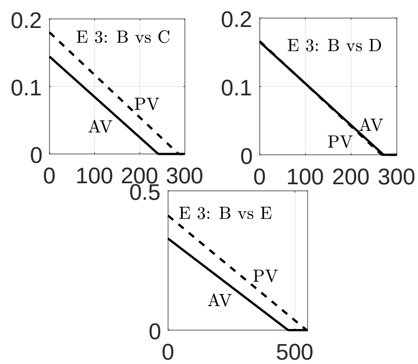


Figure 8: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates B vs C, D and E in election 3.

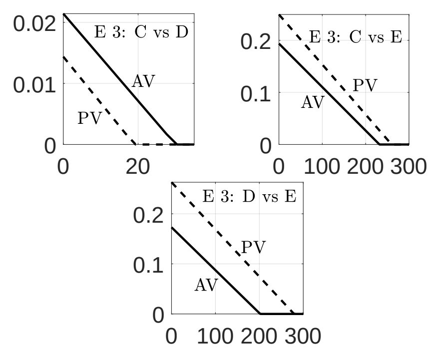


Figure 9: Robustness functions for approval voting (solid) and plurality voting (dash) between candidates C vs D and E, and D vs E in election 3.

Fig. 7 shows robustness curves for 4 races in the third TIMS election studied by Fishburn and Little (1988): candidate A vs candidates B, C, D and E. Note again the different scales in the figures. These robustness curves mostly show the pattern seen earlier: AV is robust dominant over PV when the observed margin of victory is low (A-C and A-D), but PV is robust dominant when the margin of victory is large (A-E); the A-B race deviates from the pattern because AV is slightly more robust than PV at large margin of victory.

Fig. 8 shows a similar pattern in two of the three races, and fig. 9 shows this pattern in all three races.

The pattern that is emerging is that AV robustness to uncertain voter turn out is lower than PV robustness when the margin of victory is large; at low margin of victory AV tends to be more robust than PV. Equivalently, both AV and PV robustnesses are low in highly contested races, and in these races AV tends to be more robust than PV.

In the Fishburn and Little (1988) election data, it is the low-margin races that determine the final outcome, except for the first-position race in election 3 which is won by a large margin. The B vs C race is close in elections 1 and 2; and the race for 2nd position in election 3 is close: A vs D in the PV race and C vs D in the AV race. AV is substantially more robust than PV in 3 of these 4 close elections, but not in the B vs C race in election 2, where the robustness curves cross.

Two competing factors explain these results: approval flattening and approval magnification. The former tends to reduce margins of victory in an AV election because electors can vote for more than one candidate, as explained in section 5.2. Approval magnification is the phenomenon of greater total vote counts in approval voting, which also results because electors can vote for more than one candidate in AV elections. Approval magnification tends to magnify the differences between the candidates. Approval magnification tends to enlarge margins of victory in an AV election because more votes are cast in an AV election, thus amplifying disparities between candidates. Approval flattening and approval magnification are conflicting phenomena.

In a close race the margins are small and there is not much room for approval flattening, at least among the leading candidates. This is precisely what we have seen in the data presented earlier in this section. Thus in a close race approval magnification is dominant and the margins of victory in an AV election tend to be larger than in a PV election. This tends to cause larger robustness for AV than for PV in close elections. In contrast, large margins of victory have room for approval flattening, which causes the margins of victory in a PV election to exceed those of an AV election, as we have seen earlier in this section. And this results in larger robustness for PV than for AV in races that are not close.

These two competing factors — approval flattening and approval magnification — explain the tendency for robust dominance of AV in close races, and of PV in races with large margins of victory. Furthermore, when AV is robust dominant, its robustness is low, while when PV is robust dominant its robustness is large. We stress that this conclusion is an empirically observed tendency, not a universal rule.

The significance of this tendency, to the extent that the tendency is realized, is to suggest a procedural advantage for approval voting over plurality voting in close races. Robustness to uncertainty in voter turn out is particularly important in close races. The tendency of AV to be more robust than PV in close races is therefore a procedural advantage of AV. It is not a substantive advantage based on principles of fairness or other normative issues. This is a different argument in support of AV than presented by other scholars. We note, however, that the robustness of AV in close races is low, suggesting that the procedural advantage of AV, while true as a tendency, is not particularly strong.

6 Proportional Representation: Robustness of Plurality Voting

In section 5 we discussed and compared the robustness to vote uncertainty of AV and PV in single-winner elections. There are other electoral algorithms, and we now briefly consider the robustness of PV elections leading to proportional representation, which is distinct from single-winner elections.

Rapoport, Felsenthal and Maoz (1988) discuss proportional representation of numerous parties in a legislative body. They compare plurality voting and approval voting in exit polls after an election of the Israeli General Federation of Labor (Histadrut), and note important differences in the outcomes. We will consider only the PV results, shown here in table 3.

6.1 Robustness: Formulation

We now formulate the robustness function for PV in an election leading to proportional representation. In section 6.2 we will apply the analysis to the data in table 3.

Party	Plurality Voting	
	Number of votes	Proportion of votes
Labor	275	0.668
Herut-Liberal	86	0.209
Tehiya	11	0.027
Hadash	17	0.041
Citizen Rights	18	0.044
Progressive List	3	0.007
Shinui	2	0.005
Total	412	1.000

Table 3: Exit poll plurality-voting results for Histadrut elections, from Rapoport, Felsenthal and Maoz (1988).

Let N represent the number of parties ($N = 7$ in table 3), and let $\widetilde{a} = (\widetilde{a}_1, \ldots, \widetilde{a}_N)$ denote the number of observed votes for the various parties (column 2 in table 3). The outcomes could have been different, as represented by a vector a. The uncertain deviation between $\widetilde{a}$ and a is represented by the info-gap model in eq.(7).

For any vector of voting outcomes, a, the proportion in the representative body that is assigned to the nth party is:

$$f_n = \frac{a_n}{\sum_{j=1}^{N} a_j} \tag{10}$$

The performance requirement, from the perspective of the nth party, is that its proportion in the representative body be no less than a fraction f_{c}:

$$f_n \geq f_{\mathrm{c}} \tag{11}$$

where f_{c} is any non-negative value no greater than 1.

The robustness for the nth party to satisfy this requirement is the greatest horizon of uncertainty in the voting outcomes up to which eq.(11) is satisfied for all outcome vectors a in the info-gap model:

$$\widehat{h}_n(f_{\mathrm{c}}) = \max \left\{ h : \left(\min_{a \in \mathcal{U}(h)} f_n \right) \geq f_{\mathrm{c}} \right\} \tag{12}$$

The numerical evaluation of this robustness function is explained in appendix C when the uncertainty weights satisfy eq.(8).

6.2 Robustness: Example

Figs. 10 and 11 show robustness curves, $\widehat{h}_n(f_{\mathrm{c}})$ vs f_{c}, for PV based on the data for the 7 parties in table 3. These robustness curves all display the phenomena of zeroing and trade off that we have seen in all previous examples.

Zeroing is the phenomenon that the robustness, $\widehat{h}_n(f_{\mathrm{c}})$, equals zero if the required proportional representation, f_{c}, equals the observed value. For instance, consider the robustness curve in the upper left frame in fig. 10. The curve reaches the horizontal axis (zero robustness) at the value $f_{\mathrm{c}} = 0.668$ which is the observed PV outcome for Labor. Observed outcomes have zero robustness to uncertainty in the voting results underlying those outcomes.

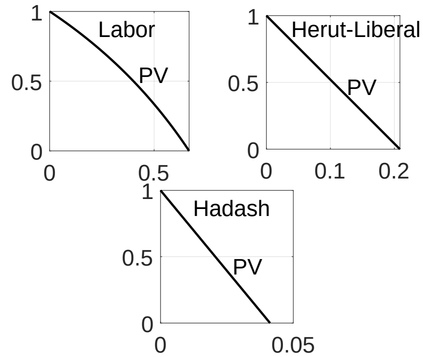


Figure 10: Robustness functions, $\widehat{h}_n(f_c)$ vs f_c, for plurality voting. $\gamma = 1$.

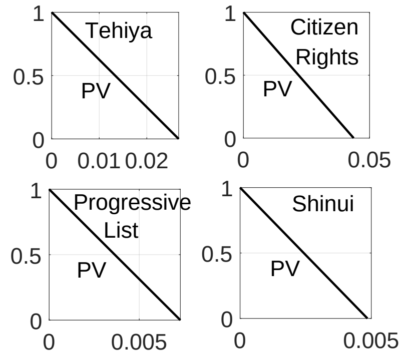


Figure 11: Robustness functions, $\widehat{h}_n(f_c)$ vs f_c, for plurality voting. $\gamma = 1$.

Trade off is the phenomenon that robustness increases as the required outcome becomes less demanding. This is expressed by the negative slopes of all the robustness curves, showing that $\widehat{h}_n(f_c)$ increases (which is desirable) as f_c decreases (which is undesirable). For example, consider again the upper left frame in fig. 10. The putative outcome for Labor is 66.8% but the robustness to uncertainty is zero for this outcome. However, the robustness curve shows that Labor's robustness to uncertainty for 50% representation ($f_c = 0.5$) is 0.32. That is, voting outcomes could vary by as much as $\pm 32\%$ and Labor would still obtain at least 50% proportional representation in a PV vote.

These conclusions demonstrate the methodology of info-gap robustness analysis and the irrevocable properties of zeroing and trade off in a PV election with proportional representation if parties.

7 Conclusion

Approval voting and plurality voting have been examined by many scholars from various normative perspectives relating to political theory and elections. In this paper we take a different approach, and examine the extent to which AV and PV are robust to uncertainty in voting outcomes. Robustness to uncertainty may itself have normative implications for electoral theory, but that has not been our concern here. Rather, we began by recognizing that voting outcomes can and do vary widely and are often highly uncertain before voting actually occurs. We then asked the question: by how much can voting outcomes change without altering the election outcomes? Whether large robustness to electoral uncertainty is desirable or not has not been our concern. Rather, we have focussed on characterizing when one voting algorithm, PV or AV, is robust dominant over the other.

In sections 3 to 5 we considered AV and PV elections between more than 2 candidates in which a single candidate wins. We found two prevailing tendencies in the evidence examined, though no irrevocable universal law.

The first tendency is that PV tends to be more robust to vote uncertainty than AV in races with large margins of victory. We explain this tendency with the concept of approval flattening, which refers to two ideas: margins of victory in single-winner elections tend to be lower in AV than in PV, and the ratio of the margin of victory to the total vote count tends to be lower in AV than in PV. We stress that margin of victory is different from robustness to uncertainty of that margin. The concept of approval flattening connects the two concepts, and explains why PV tends to greater robustness at large margins of victory.

The second tendency in single-winner elections is that AV tends to be more robust to vote uncertainty than PV at low margins of victory (just the reverse of the large-margin case). This is explained by the concept of approval magnification, which is the phenomenon of greater total vote counts in AV compared to PV. This tends to magnify the differences between the candidates in AV. Approval magnification is different from approval flattening, and they reflect different and conflicting aspects of the AV algorithm. Margins of victory are small in a close race and there is little room for approval flattening among the leading candidates. Thus in a close race the magnification is dominant and the margins of victory in an AV election are larger than in a PV election because there are more votes in an AV election. This explains the tendency for larger robustness of AV than of PV in close elections.

In section 6 we briefly considered a different electoral setting: PV voting outcomes determine the proportional representation of competing political parties. We demonstrated and discussed the implications of the zeroing and trade off properties of all info-gap robustness functions.

This paper has employed info-gap decision theory and its analysis of robustness to uncertainty. Info-gap is a strictly non-probabilistic theory for modeling and managing uncertainty. The choice between info-gap and probabilistic analyses hinges on the available evidence. Probabilistic models of vote uncertainty are more informative than info-gap models of vote uncertainty. The analyst should probably use probabilistic models *if* they are available and reliable. However, when information is insufficient to reliably support probabilistic models, then info-gap analysis of robustness to uncertainty provides a useful alternative, as demonstrated in this paper.

Robustness to uncertainty in voting outcomes has political and theoretical implications that should be explored. One might view robustness to uncertainty as undesirable because it acts as a conservative factor resisting change. Or, robustness to uncertainty might be seen as an anchor of social preference against the vagaries of voter participation and preference. These and related issues need exploration.

Our analysis has focused only on approval voting and plurality voting. Many other voting algorithms are available, and their robustness to uncertainty needs to be examined.

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A Derivation of Approval Voting Robustness Function, Eq.(6)

We now derive the robustness function for approval voting, eq.(6).

Let $m(h)$ denote the inner minimum in eq.(5), which occurs when:

$$a_i = (\widetilde{a}_i - w_i h)^+, \quad a_j = \min\{T, \ \widetilde{a}_j + w_j h\} \tag{13}$$

where we have defined a truncation function as $x^+ = x$ if $x \geq 0$ and $x^+ = 0$ otherwise.

For $h \leq (T - \widetilde{a}_j)/w_j$ and $h \leq \widetilde{a}_i/w_i$, the inner minimum is:

$$m(h) = \widetilde{a}_i - w_i h - \widetilde{a}_j - w_j h = \widetilde{a}_i - \widetilde{a}_j - h(w_i + w_j) > \delta \tag{14}$$

For $h > (T - \widetilde{a}_j)/w_j$ the inner minimum is:

$$m(h) = (\widetilde{a}_i - w_i h)^+ - T > \delta \tag{15}$$

The left hand side of this inequality is negative, while $\delta \geq 0$, so there is no non-negative solution for h. Hence the robustness is the least upper bound of solutions for h of the inequality in eq.(14), yielding:

$$\widehat{h}_{ij}(\delta) = \frac{\widetilde{a}_i - \widetilde{a}_j - \delta}{w_i + w_j} \tag{16}$$

or zero if this is negative, where $\delta \geq 0$. Note that the robustness is less than $(T - \widetilde{a}_j)/w_j$, so we needn't consider the case where $h > (T - \widetilde{a}_j)/w_j$.

B Derivation of Plurality Voting Robustness Function, Eq.(9)

Let $m(h)$ denote the inner minimum in the definition of the robustness, eq.(5), which occurs when a_i is minimal and a_j is maximal, subject to the constraints of the info-gap model in eq.(7). Recalling eq.(8) this implies:

$$a_i = (1 - \gamma h)^+ \widetilde{a}_i \tag{17}$$

Two conditions in the info-gap model constrain the maximum value of a_j.The fractional-error condition implies that $a_j \leq (1 + \gamma h)\widetilde{a}_j$. The condition $\sum_n a_n \leq T$ (deriving from the plurality condition that each voter casts at most 1 vote) imposes an upper limit on a_j that is obtained if all other candidates obtain minimal votes. Thus the two conditions in the info-gap model imply that a_j is the following minimum:

$$a_j = \min \left\{ (1 + \gamma h)\widetilde{a}_j,\ T - \sum_{n \neq j} (1 - \gamma h)^+ \widetilde{a}_n \right\} \tag{18}$$

We will now show that $(1 + \gamma h)\widetilde{a}_j$ is the minimum on the righthand side of eq.(18).

If $h \leq 1/\gamma$, then:

$$(1 + \gamma h)\widetilde{a}_j - \left(T - \sum_{n \neq j} (1 - \gamma h)^+ \widetilde{a}_n \right) = -T + (1 + \gamma h)\widetilde{a}_j + (1 - \gamma h) \sum_{n \neq j} \widetilde{a}_n \tag{19}$$

$$= \underbrace{-T + \sum_{n=1}^{N} \widetilde{a}_n}_{\leq 0} + \gamma h \underbrace{\left(\widetilde{a}_j - \sum_{n \neq j} \widetilde{a}_n \right)}_{< 0} \tag{20}$$

$$< 0 \tag{21}$$

The sum of the first two terms on the right of eq.(20) is non-positive due to the constraint in the info-gap model on the sum of the votes. The second parenthetical term on the right of eq.(20) is negative because $\widetilde{a}_i > \widetilde{a}_j$ and all votes are non-negative. The strict inequality in eq.(21) results. From this we conclude that $(1 + \gamma h)\widetilde{a}_j$ is the minimum on the righthand side of eq.(18) for all $h \leq 1/\gamma$.

Thus, for $h \leq 1/\gamma$, the inner minimum in the definition of the robustness is:

$$m(h) = (1 - \gamma h)^+ \widetilde{a}_i - (1 + \gamma h)\widetilde{a}_j = \widetilde{a}_i - \widetilde{a}_j - (\widetilde{a}_i + \widetilde{a}_j)\gamma h \tag{22}$$

The robustness is the least upper bound of h values for which $m(h) > \delta$. Equating the righthand side of eq.(22) to δ and solving for h yields:

$$\widehat{h}_{ij}(\delta) = \frac{\widetilde{a}_i - \widetilde{a}_j - \delta}{(\widetilde{a}_i + \widetilde{a}_j)\gamma} \tag{23}$$

This is less than $1/\gamma$ so we needn't consider $h > 1/\gamma$. This is eq.(9).

C Evaluating the Robustness Function for Proportional Representation by Plurality Voting

Let $m_n(h)$ denote the inner minimum in the definition of the robustness function, $\widehat{h}_n(f_c)$ in eq.(12). A plot of h vs $m_n(h)$ is the same as a plot of $\widehat{h}_n(f_c)$ vs f_c. In other words, $m_n(h)$ is the inverse of the robustness function, and knowledge of $m_n(h)$ is equivalent to knowledge of $\widehat{h}_n(f_c)$. We will derive an expression for $m_n(h)$.

From eq.(10) one can readily show that:

$$\frac{\mathrm{d}f_n}{\mathrm{d}a_n} \geq 0 \quad \text{and} \quad \frac{\mathrm{d}f_n}{\mathrm{d}a_j} \leq 0 \text{ if } j \neq n \tag{24}$$

From these relations we conclude that $m_n(h)$ is obtained when a_n is chosen as small as possible, and each a_j for $j \neq n$ is chosen as large as possible, subject to the constraints of the info-gap model, eq.(7), and employing eq.(8). Thus we find that $m_n(h)$ is obtained with:

$$a_n = (1-\gamma h)^{+}\widetilde{a}_n \tag{25}$$

$$a_j = (1+\gamma h)\widetilde{a}_j, \quad j \neq n \tag{26}$$

subject to the constraint on the sum of all N votes: $\sum_{j=1}^{N} a_j \leq T$. Thus eqs.(25) and (26) must satisfy:

$$\sum_{j=1}^{N} a_j = \min\left(T,\ (1-\gamma h)^{+}\widetilde{a}_n + (1+\gamma h)\sum_{j\neq n}\widetilde{a}_j\right) \tag{27}$$

where T is the total number of participating voters. $m_n(h)$ is obtained by substituting eq.(25) and (27) into eq.(10) yielding:

$$m_n(h) = \frac{(1-\gamma h)^{+}\widetilde{a}_n}{\min\left(T,\ (1-\gamma h)^{+}\widetilde{a}_n + (1+\gamma h)\sum_{j\neq n}\widetilde{a}_j\right)} \tag{28}$$