Lecture Notes on

The Benefit-Cost Ratio

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Source material:

• DeGarmo, E. Paul, William G. Sullivan, James A. Bontadelli and Elin M. Wicks, 1997, *Engineering Economy.* 10th ed., chapter 6, Prentice-Hall, Upper Saddle River, NJ.

• Ben-Haim, Yakov, 2010, *Info-Gap Economics: An Operational Introduction,* Palgrave-Macmillan.

• Ben-Haim, Yakov, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

Contents

[0]\lectures\Intro-Econ-DM\benefit-cost02.tex 9.5.2022

1 Incommensurate Benefits and Costs

§ **Engineering design.**

- Robotic motion.
  - Benefits:[1] stability, locational accuracy (mm).
  - Costs: components, assembly ($, or years of development).
- Airframe design.
  - Benefits: payload (kg) or speed (m/s).
  - Costs: materials and construction ($), or size ($m^3$), or weight (kg).
- Communication technology.
  - Benefits: transmission rate (bytes/s).
  - Costs: materials and manufacturing ($) or environmental damage (e.g. lost species).

§ **Infra-structure projects:**

- Roads.
  - Benefits: transportation (# people×km).
  - Costs: materials, labor ($), or political "capital" lost due to taxation.
- Parks.
  - Benefits: recreation (# people-days).
  - Costs: materials, labor, land ($).
- Sewage.
  - Benefits: public health (# saved lives).
  - Costs: materials, labor ($).
- Flood control.
  - Benefits: flood safety (# saved lives and property).
  - Costs: materials, labor ($).

§ **National defense.**

- Benefits: public security (# saved lives).
- Costs: materials, labor ($), or opportunity costs of lost health, arts, etc.

§ **The goal:**

- Given several alternative options, each technologically acceptable.
- Select one option or prioritize all the options.

§ **The problem: benefit and cost have different units.**

- The costs are (often) monetary, but the benefits (and dis-benefits) are not.
- Net worth, "benefit [e.g. mm] − cost [$]" is dimensionally inconsistent.
- Thus we cannot simply apply the capital investment and money-time relations developed previously.[2]

§ **The approach:** benefit-cost ratio (*BCR*).

Benefit-cost ratio is meaningful. E.g.:

$$\frac{\text{Benefit (e.g. \# lives or distance in km)}}{\text{Cost (\$)}} \tag{1}$$

[1]Benefit: toelet. Cost: alut.

[2]See lecture notes on Money-Time Relationships and Their Applications, money-time02.tex.

§ **Additional problems:**

- Uncertainty.
- Political considerations.
- The groups that benefit may not be the only groups that pay the cost.

§ ***BCR* commonly used to evaluate public projects.**

§ **Private vs Public projects:**[3]

- *Purpose:*
  - ◦ Private: provide goods and/or services at a profit. Maximize or satisfice profit.
  - ◦ Public: Provide services without profit; protect lives and property; provide jobs.
- *Source of capital:*
  - ◦ Private: Private investors and lenders.
  - ◦ Public: Taxation and private lenders.
- *Method of financing:*
  - ◦ Private: Individual ownership; partnerships; corporations.
  - ◦ Public: Taxation; govt bonds; user fees.
- *Nature of benefits:*
  - ◦ Private: Monetary.
  - ◦ Public: Often not monetary or difficult to monetize.
- *Measure of efficiency:*
  - ◦ Private: rate of return on capital.
  - ◦ Public: Very difficult; comparisons difficult.
- *Multiplicity of purposes:*
  - ◦ Private: Not common.
  - ◦ Public: Common. E.g.: Dam stores water, protects property, provides recreation.
- *Conflict among purposes:*
  - ◦ Private: Uncommon.
  - ◦ Public: Common. E.g.: public highways enable transport but endanger ecology.
- *Conflict of interests among stake holders:*
  - ◦ Private: Uncommon. Only one stake holder, or many with a common profit motive.
  - ◦ Public: Common. Often several or many stake holders.
- *Project duration:*
  - ◦ Private: Usually short to moderate, 5–20 years.
  - ◦ Public: Often long: 20–60 years or more.
- *Beneficiary:*
  - ◦ Private: Project owner(s) or client.
  - ◦ Public: General public.
- *Relation between beneficiaries and suppliers of capital:*
  - ◦ Private: Usually direct: same agents.
  - ◦ Public: Usually indirect or partial, via taxation.
- *Effect of politics:*
  - ◦ Private: Little to moderate.
  - ◦ Public: Frequent. Short-term tenure of decision makers, pressure groups, zoning and legal restrictions.

[3] Adapted from DeGarmo, *et al.,* table 6-1, p.240.

2 Monetizing the Benefit-Cost Ratio

2.1 Generic Monetization

§ Suppose we can monetize the benefits. E.g.: the cost (value) of a human life.

- $N =$ number of periods.
- $C_n =$ operating cost (dollars) at end of period n.
- $S =$ initial capital investment at start of period 1.
- $i_c =$ interest rate on capital.
- Large i_c (e.g. $i_c = 0.15$) means:
  - ◦ Spending \$1 now is the same as spending many \$'s later, namely \$$(1+i_c)^n 1$ at time n.
  - ◦ Spending many \$'s later is no more difficult than spending \$1 now, because later we will be richer.
- Present worth of initial investment and costs:[4]

$$C_{pw} = S + \sum_{n=1}^{N} (1+i_c)^{-n} C_n \tag{2}$$

- $B_n =$ monetized benefit (dollars) at end of period n.
- $i_b =$ discount factor on benefits, reflecting, for instance, future technological improvements or economic growth, implying enhanced future abilities.
- Large i_b (e.g. $i_b = 0.5$) means:
  - ◦ Gaining \$1 now is the same as gaining many \$'s later, namely \$$(1+i_b)^n 1$ at time n.
  - ◦ Gaining many \$'s later is no more valuable than gaining \$1 now, because later we will be richer.
  - ◦ Large economic or technological growth.
- Note different discount rates for costs and benefits because costs and benefits are substantively different. This is different from ordinary time value of money.
- Present worth of the benefits:

$$B_{pw} = \sum_{n=1}^{N} (1+i_b)^{-n} B_n \tag{3}$$

- The *BCR* is:

$$\mathit{BCR} = \frac{B_{pw}}{C_{pw}} \tag{4}$$

$$= \frac{\sum_{n=1}^{N} (1+i_b)^{-n} B_n}{S + \sum_{n=1}^{N} (1+i_c)^{-n} C_n} \tag{5}$$

- The project is worthwhile, from a benefit-cost perspective, if:

$$\mathit{BCR} > 1 \tag{6}$$

- The present worth (*PW*) of the project is:

$$\mathit{PW} = B_{pw} - C_{pw} \tag{7}$$

$$= \sum_{n=1}^{N} (1+i_b)^{-n} B_n - S - \sum_{n=1}^{N} (1+i_c)^{-n} C_n \tag{8}$$

[4]See lecture notes on Money-Time Relationships and Their Applications, money-time02.tex, for discussion of present worth.

- The project is worthwhile, from a *PW* perspective, if:

$$PW > 0 \tag{9}$$

- Question: Will eqs.(6) and (9) always:
  - Decide the same on any given project? Yes: $PW > 0$ if and only if $BCR > 1$.
  - Prioritize projects the same? Not always, as we will see.

2.2 Do *PW* and *BCR* Always Agree on Prioritization?

- Consider two projects, 1 and 2, with notation of section 2.1, p.5 and:
  - ○ $C_j = C_{pw}$ for project $j = 1$ or 2, eq.(2).
  - ○ $B_j = B_{pw}$ for project $j = 1$ or 2, eq.(3).
  - ○ $S_j = S$ for project $j = 1$ or 2.
- Suppose:

$$\text{PW}_1 = B_1 - S_1 - C_1 > B_2 - S_2 - C_2 = \text{PW}_2 \tag{10}$$

So **project 1 is *PW*-preferred.**

- But suppose:

$$S_1 + C_1 = S_2 + C_2 + D \quad \text{and} \quad B_1 = B_2 + d \quad \text{where} \quad D > 0,\ d > 0 \tag{11}$$

Question: What dilemma is embedded in these relations? Is it a BCR or a PW dilemma? Or both?

Thus:

$$\text{PW}_1 = B_2 + d - (S_2 + C_2 + D) = \text{PW}_2 + d - D \tag{12}$$

Eqs.(10) and (12) imply:

$$d > D \tag{13}$$

- Eq.(11) implies:

$$\text{BCR}_1 = \frac{B_1}{S_1 + C_1} = \frac{B_2 + d}{S_2 + C_2 + D} \tag{14}$$

- Hence **project 2 is *BCR*-preferred** if:

$$\text{BCR}_1 < \text{BCR}_2 \tag{15}$$

$$\Longleftrightarrow \quad \frac{B_2 + d}{S_2 + C_2 + D} < \frac{B_2}{S_2 + C_2} \tag{16}$$

$$\Longleftrightarrow \quad (B_2 + d)(S_2 + C_2) < B_2(S_2 + C_2 + D) \tag{17}$$

$$\Longleftrightarrow \quad d(S_2 + C_2) < B_2 D \tag{18}$$

$$\Longleftrightarrow \quad \frac{d}{D} < \frac{B_2}{S_2 + C_2} \tag{19}$$

$$\Longleftrightarrow \quad \frac{d}{D} < \text{BCR}_2 \tag{20}$$

So project 2 is *BCR*-preferred if and only if eq.(20) holds.

- Eqs.(10)–(13) and (20) can all hold, so
  ***PW* and *BCR* can disagree on prioritization of the projects.**
- **Why** is this important?
- Is one method (*PW* or *BCR*) **right** and the other **wrong?**
- How should you choose which method to use? Perhaps rank them by robustness to uncertainty.

2.3 Monetizing Human Life

§ Continue section 2.1, p.5, with this benefit function:

- $B_n = K_n L$ where:
  - $L =$ value in dollars of a human life .
  - $K_n =$ number of lives saved at end of period n.
- From eqs.(4) and (5), p.5, the *BCR* is:

$$\textit{BCR} = \frac{B_{pw}}{C_{pw}} \tag{21}$$

$$= \frac{L \sum_{n=1}^{N} (1+i_b)^{-n} K_n}{S + \sum_{n=1}^{N} (1+i_c)^{-n} C_n} \tag{22}$$

- Consider following numerical values:
  - $N = 40$ years.
  - $S =$ \$1,000,000.
  - $C_n =$ \$500,000 each year.
  - $K_n = 100$ each year.
  - $L =$ \$50,000.
  - $i_c = 0.05$. Interest rate on capital.
  - $i_b = 0.1$. Discount rate on future lives.

    What does $i_b > i_c$ imply? (Perhaps: large anticipated future population)
- The *BCR* of eq.(22) is:

$$\textit{BCR} = \frac{LK \sum_{n=1}^{N} (1+i_b)^{-n}}{S + C \sum_{n=1}^{N} (1+i_c)^{-n}} \tag{23}$$

$$= \frac{LK \frac{1-(1+i_b)^{-N}}{i_b}}{S + C \frac{1-(1+i_c)^{-N}}{i_c}} \tag{24}$$

$$= \frac{LK \delta_f(i_b)}{S + C\delta_f(i_c)} \tag{25}$$

Where $\delta_f(i)$ is a "discount function:"

$$\delta_f(i) = \frac{1-(1+i)^{-N}}{i} \tag{26}$$

- We find:
  - $\delta_f(i_b) = 9.7791$, $\delta_f(i_c) = 17.1591$, *BCR* $= 5.1041$.
- **Project is highly justified** based on the *BCR* analysis:

  \$5.1 of present-worth benefit for each \$1 of present-worth cost.

2.4 Monetizing Human Life with Uncertain L

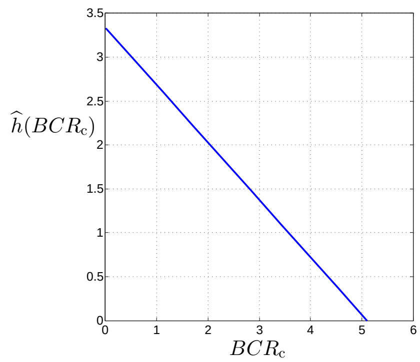

Figure 1: Robustness curve, eq.(32), with parameter values of section 2.3 and $s_L = 0.3\widetilde{L}$ =\$15,000.

§ Continue section 2.3, p.8, with uncertain L:

$$\mathcal{U}(h) = \left\{ L : \left| \frac{L - \widetilde{L}}{s_L} \right| \le h \right\}, \quad h \ge 0 \tag{27}$$

- Require: *BCR*$(L) \ge BCR_c$.
- Robustness:

$$\widehat{h}(BCR_c) = \max \left\{ h : \left(\min_{L \in \mathcal{U}(h)} \textit{BCR}(L) \right) \ge \textit{BCR}_c \right\} \tag{28}$$

- Inner minimum, $m(h)$, occurs at $L = \widetilde{L} - s_L h$. From eq.(25), p.8:

$$m(h) = \underbrace{\frac{K\delta_f(i_b)}{S + C\delta_f(i_c)}}_{Q = BCR(\widetilde{L})/\widetilde{L}} (\widetilde{L} - s_L h) \tag{29}$$

- Equate this to BCR_c and solve for h to find robustness:

$$\widehat{h}(BCR_c) = \frac{Q\widetilde{L} - BCR_c}{s_L Q} \tag{30}$$

$$= \frac{\textit{BCR}(\widetilde{L}) - BCR_c}{s_L \textit{BCR}(\widetilde{L})/\widetilde{L}} \tag{31}$$

$$= \frac{\widetilde{L}}{s_L} \left(1 - \frac{BCR_c}{\textit{BCR}(\widetilde{L})} \right) \quad \text{or zero if this is negative} \tag{32}$$

- Zeroing: $\widehat{h}[\textit{BCR}(\widetilde{L})] = 0$.
- Trade off: slope = $-\dfrac{1}{s_L Q} = -\dfrac{\widetilde{L}}{s_L \textit{BCR}(\widetilde{L})}$.

  Question: Do we want small or large negative slope? See fig. 2, p.10.
  - ◦ Looking from the top: $\widehat{h}$ decreases fast; looks **bad.**
  - ◦ Looking from the bottom: $\widehat{h}$ increases fast; looks **good.**

  Steep slope: low **cost of robustness:** is that **good** or **bad**?

  Low cost of robustness if $\widetilde{L} \gg s_L$ (low uncertainty) or if *BCR*$(\widetilde{L})$ is small (low value).
- See fig. 1 with numerical values from section 2.3, p.8, and $s_L = 0.3\widetilde{L}$ =\$15,000.
- Moderate robustness at moderate BCR_c, fig. 1:

- ◦ **Question:** Could you responsible sell this program with a BCR of 4 or 5?
- ◦ $\widehat{h}(BCR_{\rm c} = 1) = 2.7$.
- ◦ $\widehat{h}(BCR_{\rm c} = 2) = 2.0$.

- **The project looks *BCR*-plausible,** even with uncertainty in L.

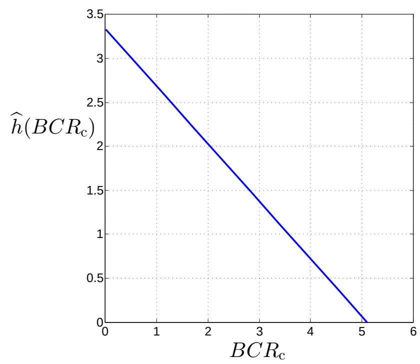

Figure 2: Robustness curve, eq.(32), with parameter values of section 2.3 and $s_L = 0.3\widetilde{L} =$\$15,000.

2.5 Monetizing Human Life with Uncertain L and i_b

§ Continue section 2.3, p.8, with uncertain L and i_b. Assume that i_b is constant but uncertain:

$$\mathcal{U}(h) = \left\{ L, i_b : \left| \frac{L - \widetilde{L}}{s_L} \right| \le h, \ i_b > -1, \ \left| \frac{i_b - \widetilde{i}_b}{s_i} \right| \le h \right\}, \quad h \ge 0 \tag{33}$$

- Require: *BCR*$(L, i_b) \ge BCR_{\rm c}$.
- Robustness:

$$\widehat{h}(BCR_{\rm c}) = \max \left\{ h : \ \left(\min_{L, i_b \in \mathcal{U}(h)} BCR(L, i_b) \right) \ge BCR_c \right\} \tag{34}$$

- From eq.(23), p.8, inner minimum, $m(h)$, occurs at:
  - ∘ $L = \widetilde{L} - s_L h$.
  - ∘ $i_b = \widetilde{i}_b + s_i h$ (**Why?** See eq.(22), p.8.) if $\widetilde{L} - s_L h \ge 0$ (**Why?**) or $h \le \widetilde{L}/s_L$.

$$m(h) = \frac{K \sum_{n=1}^{N} (1 + \widetilde{i}_b + s_i h)^{-n}}{S + C \sum_{n=1}^{N} (1 + i_c)^{-n}} \, (\widetilde{L} - s_L h) \tag{35}$$

$$= \frac{K \frac{1 - (1 + \widetilde{i}_b + s_i h)^{-N}}{\widetilde{i}_b + s_i h}}{S + C \frac{1 - (1 + i_c)^{-N}}{i_c}} \, (\widetilde{L} - s_L h) \tag{36}$$

$$= \frac{K \delta_f(\widetilde{i}_b + s_i h)}{S + C \delta_f(i_c)} \, (\widetilde{L} - s_L h) \quad \text{for } h \le \widetilde{L}/s_L \tag{37}$$

- $m(h)$ is the inverse of the robustness:

$$m(h) = BCR_{\rm c} \quad \Longleftrightarrow \quad \widehat{h}(BCR_{\rm c}) = h \tag{38}$$

- See fig. 4 with numerical values from section 2.3, p.8, and $s_L = 0.3\widetilde{L}$ =$15,000 and $s_i = 0.3\widetilde{i}_b = 0.03$.
- Moderate robustness at moderate $BCR_{\rm c}$, fig. 4:
  - ∘ $\widehat{h}(BCR_{\rm c} = 1) = 2.3$.
  - ∘ $\widehat{h}(BCR_{\rm c} = 2) = 1.5$.
- **The project still looks *BCR*-plausible,** even with uncertainty in L and i_b.
  - ∘ Only slightly less robust than section 2.4, fig. 3. Intercepts are the same:
    Horizontal intercept at $BCR_{\rm c} =$ *BCR*$(\widetilde{L}, \widetilde{i}_b) = 5.1041$.
    Vertical intercept at $h = \widetilde{L}/s_L$ = 1/0.3 = 3.33. ■

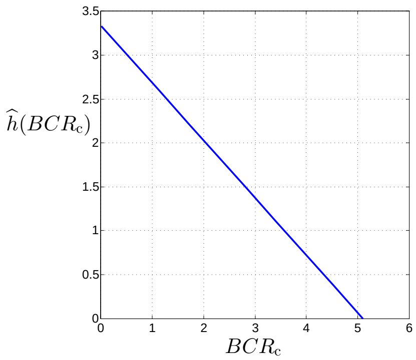

Figure 3: Robustness curve, eq.(32), with parameter values of section 2.3 and $s_L = 0.3\widetilde{L}$ =$15,000. Same as fig. 1, p.9.

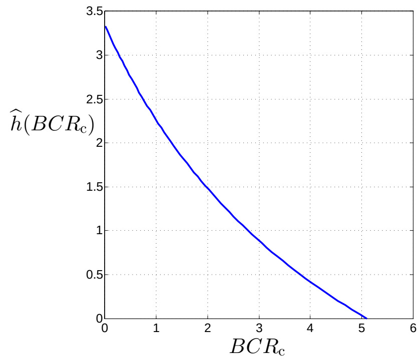

Figure 4: Robustness curve, eq.(37), with parameter values of section 2.3 and $s_L = 0.3\widetilde{L}$ =$15,000 and $s_i = 0.3\widetilde{i}_b = 0.03$.

2.6 Monetizing Human Life with Uncertain L, i_b, K and C

§ Continue with BCR from eq.(22), p.8.

§ Continue section 2.3, p.8, with uncertain L, i_b, K and C, where i_b is constant but uncertain:

$$\mathcal{U}(h) = \left\{ L, i_b, K, C : \left| \frac{L - \widetilde{L}}{s_L} \right| \le h,\ i_b > -1,\ \left| \frac{i_b - \widetilde{i}_b}{s_i} \right| \le h,\ \left| \frac{K - \widetilde{K}}{s_K} \right| \le h,\ \left| \frac{C - \widetilde{C}}{s_C} \right| \le h,\ \right\}, \quad h \ge 0 \tag{39}$$

- Require: *BCR*$(L, i_b, K, C) \ge BCR_{\rm c}$.
- Robustness:

$$\widehat{h}(BCR_{\rm c}) = \max \left\{ h : \left(\min_{L, i_b, K, C \in \mathcal{U}(h)} \textit{BCR}(L, i_b, K, C) \right) \ge \textit{BCR}_c \right\} \tag{40}$$

- From eq.(23), p.8, inner minimum, $m(h)$, for $h \le \min(\widetilde{L}/s_L,\ \widetilde{K}/s_K)$, occurs at:
  - $L = \widetilde{L} - s_L h$. $K = \widetilde{K} - s_K h$. $C = \widetilde{C} + s_C h$.
  - $i_b = \widetilde{i}_b + s_i h$.

$$m(h) = \frac{\sum_{n=1}^{N} (1 + \widetilde{i}_b + s_i h)^{-n}}{S + (\widetilde{C} + s_C h) \sum_{n=1}^{N} (1 + i_c)^{-n}} (\widetilde{L} - s_L h)(\widetilde{K} - s_K h) \tag{41}$$

$$= \frac{\frac{1 - (1 + \widetilde{i}_b + s_i h)^{-N}}{\widetilde{i}_b + s_i h}}{S + (\widetilde{C} + s_C h) \frac{1 - (1 + i_c)^{-N}}{i_c}} (\widetilde{L} - s_L h)(\widetilde{K} - s_K h) \tag{42}$$

$$= \frac{\delta_f(\widetilde{i}_b + s_i h)}{S + (\widetilde{C} + s_C h)\delta_f(i_c)} (\widetilde{L} - s_L h)(\widetilde{K} - s_K h) \quad \text{for } h \le \min(\widetilde{L}/s_L,\ \widetilde{K}/s_K) \tag{43}$$

- $m(h)$ is the inverse of the robustness:

$$m(h) = BCR_{\rm c} \quad \Longrightarrow \quad \widehat{h}(BCR_{\rm c}) = h \tag{44}$$

- See fig. 7 with numerical values from section 2.3, p.8, and $s_L = 0.3\widetilde{L}$ =\$15,000, $s_i = 0.1\widetilde{i}_b =$ 0.03, $s_K = 0.3\widetilde{K}$ =30, $s_C = 0.1\widetilde{C}$ =\$50,000.
- Low robustness at moderate $BCR_{\rm c}$, fig 7:
  - $\widehat{h}(BCR_{\rm c} = 1) = 1.5$.
  - $\widehat{h}(BCR_{\rm c} = 2) = 0.91$.
- **The project looks barely *BCR*-plausible** with uncertainty in L, i_b, K and C.
  - Less robust than section 2.4 (fig 5) or section 2.5 (fig 6). Intercepts are the same:
    Horizontal intercept at $BCR_{\rm c} =$ *BCR*$(\widetilde{L}, \widetilde{i}_b) =$ 5.1041.
    Vertical intercept at $h = \min(\widetilde{L}/s_L, \widetilde{K}/s_K)$ = 1/0.3 = 3.33.

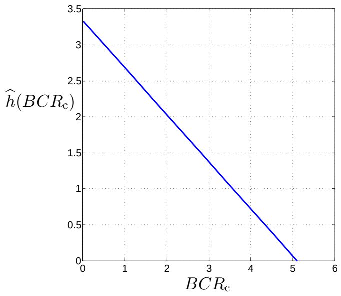

Figure 5: Robustness curve, eq.(32), with parameter values of section 2.3 and $s_L = 0.3\widetilde{L}$ =\$15,000. Same as fig. 1, p.9.

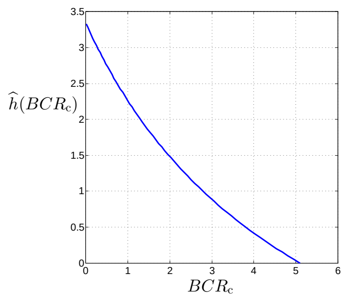

Figure 6: Robustness curve, eq.(37), with parameter values of section 2.3 and $s_L = 0.3\widetilde{L}$ =\$15,000 and $s_i = 0.3\widetilde{i}_b = 0.03$. Same as fig. 4, p.12.

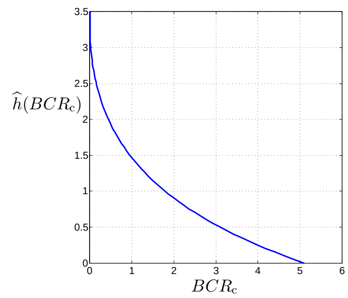

Figure 7: Robustness curve, eq.(43), with parameter values of section 2.3 and $s_L = 0.3\widetilde{L}$ =\$15,000, $s_i = 0.1\widetilde{i}_b = 0.03$, $s_K = 0.1\,\widetilde{K} = 30$, $s_C = 0.1\,\widetilde{C}$ =\$50,000.

2.7 Constant But Uncertain Interest Rates i_b and i_c

§ Continue section 2.3, p.8, with constant but uncertain interest rates.

- *BCR* of eq.(5), p.5, with constant B and C:

$$\text{\textit{BCR}} = \frac{B \sum_{n=1}^{N} (1+i_b)^{-n}}{S + C \sum_{n=1}^{N} (1+i_c)^{-n}} \tag{45}$$

$$= \frac{B \frac{1-(1+i_b)^{-N}}{i_b}}{S + C \frac{1-(1+i_c)^{-N}}{i_c}} \tag{46}$$

$$= \frac{B\delta_f(i_b)}{S + C\delta_f(i_c)}, \quad \delta_f(i) \text{ defined in eq.(26), p.8} \tag{47}$$

- Interest rate for benefits, i_b, highly uncertain. Diverse criteria for choosing i_b:[5]
  - ○ Opportunity cost to government.
  - ○ Opportunity cost to tax payers.
  - ○ Subjective discount rate on future population growth or technological development.
- Interest rate for costs, i_c, uncertain:
  - ○ Future cost of money uncertain.
  - ○ Future financing opportunities uncertain.
- Numerical values:
  - ○ $B =$\$5,000,000.
  - ○ $C =$\$500,000.
  - ○ $S =$\$1,000,000.
  - ○ $N = 40$ years.
- *BCR* increases as i_b decreases (**Why?**), strongly for $i_b < 0.1$, fig. 8.
  - ○ Small i_b implies future benefits are nearly as important as present benefits.
  - ○ Large i_b ignores (discounts) the future.
  - ○ Implication of fig. 8:
    including future benefits (small i_b) makes the present more attractive (large *BCR*).
- *BCR* increases as i_c increases (**Why** different from i_b?), fig. 9.
  - ○ Large i_c ignores (discounts) future costs.
  - ○ Small i_c implies future costs are nearly as important as present costs.
  - ○ Implication of fig. 9:
    ignoring future costs (large i_c) makes the present more attractive (large *BCR*).

[5] DeGarmo *et al.,* p.246.

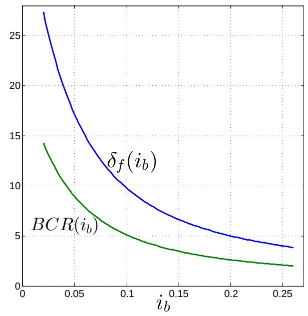

Figure 8: *BCR*, eq.(47), and $\delta_f(i_b)$ vs i_b, with $i_c = 0.05$, $\delta_f(i_c) = 17.16$.

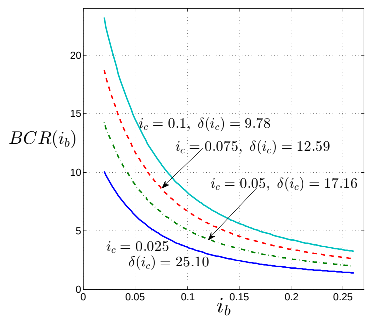

Figure 9: *BCR*, eq.(47), and $\delta_f(i_b)$ vs i_b, with 4 i_c's.

2.8 Benefits, Dis-Benefits and Conflicting Interests

- Benefits and dis-benefits:
  - Increased stiffness of a beam by adding ribs also increases the weight.
    Enhancing the reliability may reduce the allowable payload.
    The reliability engineer's benefits are the flight engineer's dis-benefits.
  - Highways sometimes disturb habitats and damage ecologies.
    The motorists' benefits are the naturalists' dis-benefits.
  - Increased product life delays the opportunity for up-grade.
    The planner's benefit is the innovator's dis-benefit.
- Present worth of benefits, B_n, and dis-benefits, D_n, adapting from eq.(3), p.5:

$$B_{pw} = \sum_{n=1}^{N} (1+i_b)^{-n}(B_n - D_n) \tag{48}$$

- *BCR*, from eqs.(2), (4) and (48):

$$\textit{BCR} = \frac{B_{pw}}{C_{pw}} \tag{49}$$

$$= \frac{\sum_{n=1}^{N}(1+i_b)^{-n}(B_n - D_n)}{S + \sum_{n=1}^{N}(1+i_c)^{-n} C_n} \tag{50}$$

Special case: B_n, D_n and C_n are constant, so eq.(50) is:

$$\textit{BCR} = \frac{(B-D)\sum_{n=1}^{N}(1+i_b)^{-n}}{S + C\sum_{n=1}^{N}(1+i_c)^{-n}} \tag{51}$$

$$= \frac{(B-D)\frac{1-(1+i_b)^{-N}}{i_b}}{S + C\frac{1-(1+i_c)^{-N}}{i_c}} \tag{52}$$

$$= \frac{(B-D)\delta_f(i_b)}{S + C\delta_f(i_c)}, \quad \delta_f(i) \text{ defined in eq.(26), p.8} \tag{53}$$

- Uncertain dis-benefits:

$$\mathcal{U}(h) = \left\{ D : \left| \frac{D - \widetilde{D}}{s_D} \right| \le h \right\}, \quad h \ge 0 \tag{54}$$

- Robustness for requirement *BCR*$(D) \ge BCR_{\rm c}$:

$$\widehat{h}(BCR_{\rm c}) = \max \left\{ h : \left(\min_{D\in\mathcal{U}(h)} \textit{BCR}(D) \right) \ge \textit{BCR}_c \right\} \tag{55}$$

- Inner minimum, $m(h)$, occurs at $D = \widetilde{D} + s_D h$:

$$m(h) = \frac{(B - \widetilde{D} - s_D h)\delta_f(i_b)}{S + C\delta_f(i_c)} \tag{56}$$

$$= \textit{BCR}(\widetilde{D}) - \frac{s_D \delta_f(i_b)}{S + C\delta_f(i_c)} h \tag{57}$$

- Equate eq.(57) to $BCR_{\rm c}$ and solve for h to find robustness:

$$\textit{BCR}(\widetilde{D}) - \frac{s_D \delta_f(i_b)}{S + C\delta_f(i_c)} h = BCR_{\rm c} \quad \Longrightarrow \quad \widehat{h}(BCR_{\rm c}) = \frac{(\textit{BCR}(\widetilde{D}) - BCR_{\rm c})(S + C\delta_f(i_c))}{s_D \delta_f(i_b)} \tag{58}$$

- Values of B, C, S and N from section 2.7, p.15, with $i_c = 0.05$, $s_D = 0.3\widetilde{D}$. Fig. 10.
- Horizontal intercepts (zeroing):
  - *BCR*$(\widetilde{D} = \$2M, i_b = 0.1) = 3.06 > 2.43 =$ *BCR*$(\widetilde{D} = \$1.5M, i_b = 0.15)$:
  - In this case, lower discounting ($i_b = 0.1$) nominally outweighs larger dis-benefit ($\widetilde{D}$ =\$2M).
- Cost of robustness (slopes):
  - slope$(\widetilde{D} = \$2M, i_b = 0.1) = -1.63 > -3.21 =$ slope$(\widetilde{D} = \$1.5M, i_b = 0.15)$.
  - Lower cost of robustness with $\widetilde{D} = \$1.5M$ due to lower uncertainty: $s_D \propto \widetilde{D}$.
- Preference reversal: trade off between dis-benefit and discounting depends on BCR_c.

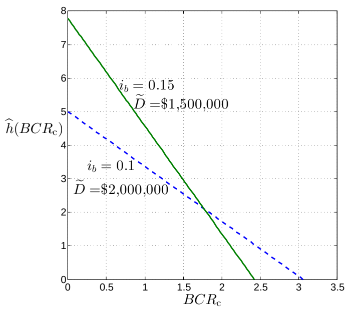

Figure 10: Robustness curve, eq.(58).

3 Using the *BCR* with Incommensurate Benefits and Costs

3.1 Robotic Position Accuracy

- Robotic arm with positional accuracy d [mm].
- Small d better than large d: number of available tasks increases as d decreases, table 1.
- Small d is more expensive than large d, table 1.

d [mm]	# tasks	eq.(59)	Price ($\$10^5$)	eq.(60)
1	50	50.0	10	10
2	25	25.0	5	5.0
3	12	12.5	3.4	3.3
4	6	6.25	2.5	2.5

Table 1: Data for section 3.1.

- ◦ Benefit function, $B(d)$, col. 3, table 1:

$$B(d) = B_0 \mathrm{e}^{-\lambda d}, \quad B_0 = 100 \text{ [\# of tasks]}, \quad \lambda = 0.693 \tag{59}$$

- ◦ Price function, $S(d)$, col. 5, table 1:

$$S(d) = S_0/d, \quad S_0 = \$10^6 \tag{60}$$

- $C(d) =$ maintenance cost at end of each year = $\varepsilon S(d)$. We will use $\varepsilon = 0.15$.
- $N =$ life of robot = 5 years.
- $i_c =$ interest rate or MARR = 0.05.
- **The task:** specify positional accuracy that's worth the money.
- *PW* of initial cost and maintenance, eq.(2), p.5:

$$C_{pw}(d) = S(d) + \sum_{n=1}^{N} (1+i_c)^{-n} C(d) \tag{61}$$

$$= S(d)\left(1 + \varepsilon \sum_{n=1}^{N} (1+i_c)^{-n}\right) \tag{62}$$

$$= S(d)\left(1 + \varepsilon \frac{1-(1+i_c)^{-N}}{i_c}\right) \tag{63}$$

$$= S(d)\,(1 + \varepsilon \delta_f(i_c)) \tag{64}$$

$\delta_f(i_c) = 4.33$ so $1 + \varepsilon\delta_f(i_c) = 1.65$ so $C_{pw}(d) = 1.65 S(d)$.

- **The problem,** fig. 11, p.20:
  - ◦ Benefit improves ($B(d)$ rises) and cost rises $C_{pw}(d)$ as accuracy improves (d falls).
  - ◦ The usual calculation of worth is $B - C$, but this is now **dimensionally inconsistent.**

- **The solution:** consider benefit per dollar, the *BCR* in units [# of tasks/$]:

$$\textit{BCR}(d) = \frac{B(d)}{C_{pw}(d)} \tag{65}$$

$$= \frac{B(d)}{S(d)(1+\varepsilon\delta_f(i_c))} \tag{66}$$

$$= \frac{B_0 e^{-\lambda d}}{S_0/d} \frac{1}{1+\varepsilon\delta_f(i_c)} \tag{67}$$

$$= \frac{B_0 d e^{-\lambda d}}{S_0} \frac{1}{1+\varepsilon\delta_f(i_c)} \tag{68}$$

- Using the *BCR*, fig. 12, p.20:
  - ◦ *BCR*(d) maximal and fairly constant for $1 \le d \le 2$ [mm].
    Range of best economic efficiency.
  - ◦ *BCR*(d) falls as d goes: $1 \mapsto 0$.
    Range of diminishing economic efficiency.
  - ◦ *BCR*(d) falls as d goes: $2 \mapsto 4$.
    Range of diminishing economic efficiency.

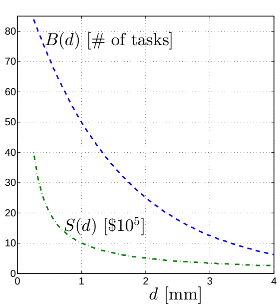

Figure 11: Benefit and initial cost vs positional accuracy.

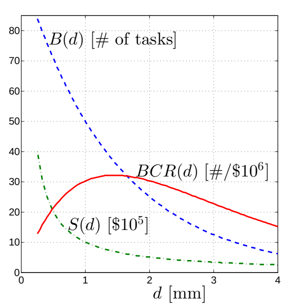

Figure 12: *BCR* vs positional accuracy, d, eq.(68), with benefit and initial cost functions.

- Note: Economic efficiency isn't everything.
  If you need spatial accuracy of, say, 0.3 mm,
  or **if you need** great versatility, $B(0.3) = 81$ tasks,
  then you need $d = 0.3$ mm despite the economic inefficiency.

3.2 Robotic Position Accuracy: Comparing 3 Designs

- Continue section 3.1, p.19.
- Compare three different designs, table 2, eqs.(69)–(74) and figs. 13 and 14, p.22.

d [mm]	$B_1(d)$	$S_1(d)$ (\$$10^5$)	$B_2(d)$	$S_2(d)$ (\$$10^5$)	$B_3(d)$	$S_3(d)$ (\$$10^5$)
1	50	10	34	9	67	9
2	25	5	26	7	45	7
3	12.5	3.4	18	5	23	5
4	6.25	2.5	10	3	1	3

Table 2: Data for section 3.2.

Design 1: $$B_1(d) = B_0 e^{-\lambda d}, \quad B_0 = 100 \text{ [\# of tasks]}, \quad \lambda = 0.693 \tag{69}$$

$$S_1(d) = S_0/d, \quad S_0 = \$10^6 \tag{70}$$

Design 2: $$B_2(d) = -m_2 d + g_2, \quad m_2 = -8, \quad g_2 = 42 \tag{71}$$

$$S_2(d) = -a_2 d + b_2, \quad a_2 = -2, \quad b_2 = 1 \tag{72}$$

Design 3: $$B_3(d) = -m_3 d + g_3, \quad m_3 = -22, \quad g_3 = 89 \tag{73}$$

$$S_2(d) = -a_3 d + b_3, \quad a_3 = a_2 = -2, \quad b_3 = b_2 = 1 \tag{74}$$

- Design 1: Same as section 3.1, p.19.
  - Good accuracy at low d, fig. 13.
  - High cost at low d, fig. 14.
- Design 2:
  - Better accuracy than Design 1 at large d. Worse accuracy than Design 1 at small d.
  - Higher cost than Design 1 at large d. Lower cost than Design 1 at small d.
- Design 3:
  - Better accuracy than Design 1 at all d.
  - Higher cost than Design 1 at large d. Lower cost than Design 1 at small d.
- *BCR*$_j$ for design j, from eq.(66), p.20:

$$BCR_j(d) = \frac{B_j(d)}{S_j(d)(1 + \varepsilon\delta_f(i_c))} \tag{75}$$

- *BCR*, fig. 15:
  - Design 3: Best economic efficiency (*BCR*) for $d < 3.3$.
  - Design 3: Worst economic efficiency (*BCR*) for $d > 3.3$.
  - Design 2: Best economic efficiency (*BCR*) for $d > 3.3$.

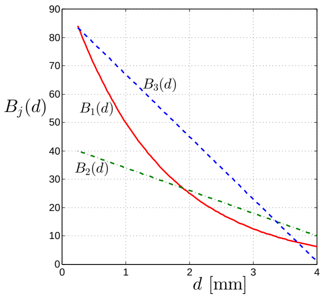

Figure 13: Benefit functions.

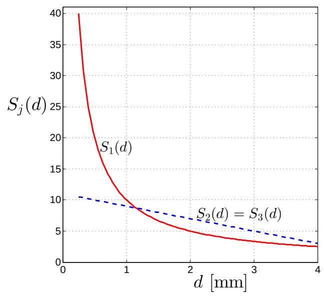

Figure 14: Initial cost functions.

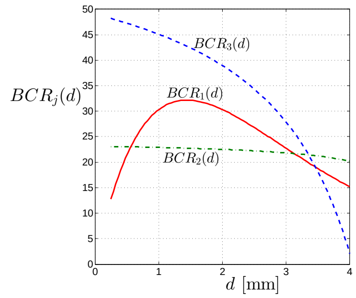

Figure 15: *BCR* vs positional accuracy, d, eq.(75), with 3 benefit and initial cost functions.

3.3 Robotic Position Accuracy with Uncertain Benefit

- Return to section 3.1, p.19 and consider uncertain $B(d)$.
- The *BCR*, eq.(66), p.20, is:

$$\textit{BCR} = \frac{B(d)}{S(d)(1+\varepsilon\delta_f(i_c))} \tag{76}$$

$i_c = 0.05$, $N = 5$, $\varepsilon = 0.15$, $1 + \varepsilon\delta_f(i_c) = 1.65$. From eq.(60):

$$S(d) = S_0/d, \quad S_0 = \$10^6 \tag{77}$$

and, from eq.(59), our uncertain estimate of the benefit function is:

$$\widetilde{B}(d) = B_0 \mathrm{e}^{-\lambda d}, \quad B_0 = 100 \ [\text{\# of tasks}], \quad \lambda = 0.693 \tag{78}$$

- However, we don't know how much $\widetilde{B}(d)$ errs, so we use a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ B(d) : \ \left| \frac{B(d) - \widetilde{B}(d)}{\widetilde{B}(d)} \right| \le h \right\}, \quad h \ge 0 \tag{79}$$

- We require that the *BCR* be no less than a critical value, BCR_{c}:

$$\textit{BCR}(B, d) \ge BCR_{\mathrm{c}} \tag{80}$$

- The robustness is the greatest tolerable horizon of uncertainty:

$$\widehat{h}(BCR_{\mathrm{c}}, d) = \max \left\{ h : \ \left(\min_{B \in \mathcal{U}(h)} \textit{BCR}(B, d) \right) \ge \textit{BCR}_c \right\} \tag{81}$$

- The inner minimum, $m(h)$, occurs when $B(d)$ is as small as possible:

$$m(h) = \frac{(1-h)\widetilde{B}(d)}{S(d)(1+\varepsilon\delta_f(i_c))} \tag{82}$$

$$= (1-h)\textit{BCR}(\widetilde{B}, d) \tag{83}$$

- Equate $m(h)$ to BCR_{c} and solve for h:

$$(1-h)\textit{BCR}(\widetilde{B}, d) = BCR_{\mathrm{c}} \quad \Longrightarrow \tag{84}$$

$$\widehat{h}(BCR_{\mathrm{c}}, d) = 1 - \frac{BCR_{\mathrm{c}}}{\textit{BCR}(\widetilde{B}, d)} \tag{85}$$

$$= 1 - \frac{S_0 \mathrm{e}^{\lambda d}}{B_0 d}(1+\varepsilon\delta_f(i_c))BCR_{\mathrm{c}} \tag{86}$$

or zero if this is negative.

- Robustness vs critical BCR, fig. 16, for 3 different positional accuracies d:
  - ∘ Zeroing: $\widehat{h}(BCR_{\rm c}) = 0$ at $BCR_{\rm c} =$ *BCR*$(\widetilde{B})$ = value in fig. 12, p.20.
    This determines the order of the curves.
  - ∘ Trade off: robustness vs critical BCR that can be achieved.
    $\widehat{h}(BCR_{\rm c} = 16$ tasks$/\$10^6,\ d = 1.3) = 0.5$.
- Robustness vs positional accuracy, fig. 17, for 3 critical BCRs.
  - ∘ $d = 1.4$ [mm] is most robust positional accuracy.
  - ∘ 30 tasks/\$$10^6$: very low robustness; probably infeasible.
  - ∘ 10 or 20 tasks/\$$10^6$: low/modest robustness at $d = 1.4$ [mm]; may be feasible.

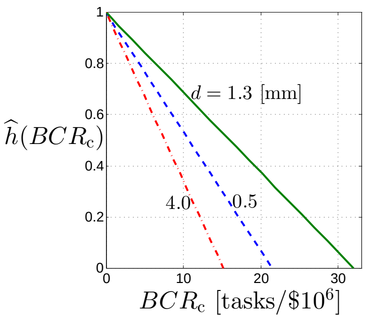

Figure 16: Robustness vs critical # of tasks, eq.(86), for 3 positional accuracies d.

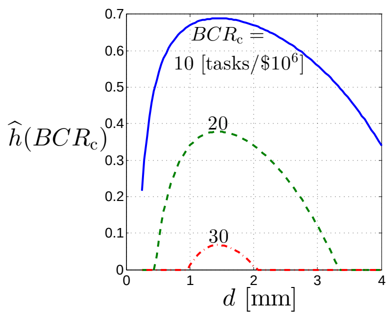

Figure 17: Robustness vs positional accuracy, eq.(86), for 3 $BCR_{\rm c}$'s.

3.4 Discounting Future Non-Monetary Benefit: Sorties of a Drone

- Question:
  - ◦ We know how to discount the future value of money: time value of money.
  - ◦ How to discount the future value of non-monetary benefit?

- Consider an intelligence-gathering drone:
  - ◦ $N =$ life = 5 [years].
  - ◦ $B_n =$ benefit in year n, E.g. = number of sorties in nth year = 100.
  - ◦ $C_n =$ maintenance cost at end of nth year = \$2,000.
  - ◦ $S =$ initial cost of drone = \$10,000.
- *PW* of investment and maintenance, eq.(2), p.5:

$$C_{pw} = S + \sum_{n=1}^{N} (1 + i_c)^{-n} C_n \tag{87}$$

$i_c =$ interest rate = 0.05.
- Discounting the future:
  - ◦ $i_b =$ discount rate, expressing reduced importance of future benefit (e.g. sorties) due to:
    - — Alternative future intelligence-gathering methods.
    - — Less dangerous security environment, reducing need for drones.
    - — More concealed security threats, reducing utility of drones.
  - ◦ We will use $i_b = 0.15$.
  - ◦ i_b may be quite uncertain, due to uncertain future technology or security environment.
  - ◦ We will info-gap i_b in section 3.5, p. 28.
- *PW* of benefits, eq.(3), p.5:

$$B_{pw} = \sum_{n=1}^{N} (1 + i_b)^{-n} B_n \tag{88}$$

  - ◦ Note: Single benefit, B_n, in each period. This is a simplification.
  - ◦ However, there can be different benefits, of different importance, over time: Tactical, strategic or political intelligence; etc.
- *BCR*, eqs.(4) and (5), p.5:

$$\textit{BCR} = \frac{B_{pw}}{C_{pw}} \tag{89}$$

$$= \frac{\sum_{n=1}^{N} (1 + i_b)^{-n} B_n}{S + \sum_{n=1}^{N} (1 + i_c)^{-n} C_n} \tag{90}$$

- If:

$$B = B_n, \quad C = C_n \tag{91}$$

then:

$$\textit{BCR} = \frac{\frac{1-(1+i_b)^{-N}}{i_b} B}{S + \frac{1-(1+i_c)^{-N}}{i_c} C} \tag{92}$$

$$= \frac{\delta_f(i_b) B}{S + \delta_f(i_c) C} \tag{93}$$

- With eq.(93), and for $i_c = 0.05$, $i_b = 0.15$, etc., we find:

$$\delta_f(i_b) = 3.3522, \quad \delta_f(i_c) = 4.3295, \quad \textit{BCR} = 0.0180 \text{ [sorties/\$]} \tag{94}$$

∘ One time-discounted sortie costs $1/\textit{BCR} = 1/0.0180 = \$55.56$/sortie.
∘ BCR increases linearly as B (# of sorties/year) increases, eq.(93), fig. 18, p.26.
∘ BCR decreases non-linearly as i_b (discount rate for future benefit) increases, fig. 19, p.26.
∘ Both B and i_b are uncertain.

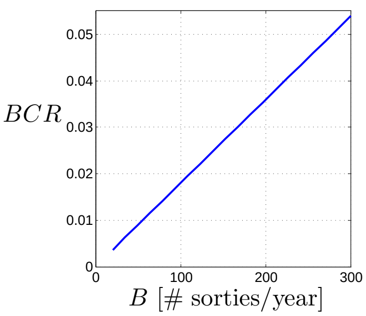

Figure 18: BCR vs # of sorties/year, eq.(86). $i_b = 0.15$.

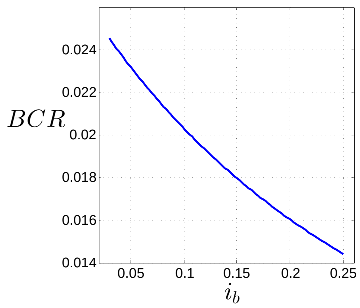

Figure 19: BCR vs benefit discount rate, eq.(86). $B = 100$.

• Compare eq.(94) with shorter duration and proportionately lower initial investment:
∘ $N =$ life = 2 [years].
∘ $B_n =$ benefit in year n, E.g. = number of sorties in nth year = 100.
∘ $C_n =$ maintenance cost at end of nth year = \$2,000.
∘ $S =$ initial cost of drone = \$4,000.
∘ $i_b = 0.15$, $i_c = 0.05$.
∘ With eq.(93) we find:

$$\delta_f(i_b) = 1.6257, \quad \delta_f(i_c) = 1.8594, \quad \textit{BCR} = 0.0211 \text{ [sorties/\$]} \tag{95}$$

∘ One time-discounted sortie costs $1/\textit{BCR} = 1/0.0211 = \$47.48$/sortie.
∘ This is lower (better) cost/sortie than eq.(94), \$55.56/sortie, because the higher cost at $N = 5$ is spread over discounted (lower) benefits.
∘ This raises the idea of **discounted fair price:** An initial cost function $S(N)$ for which *BCR*(N) is constant and equals *BCR*$_{\rm ref}$, a constant given reference value. For each N, solve this relation for $S(N)$, using also eq.(93), p.25:

$$\textit{BCR}_{\rm ref} = \textit{BCR}(N, S(N)) \tag{96}$$

$$= \frac{\delta_f(i_b, N)B}{S(N) + \delta_f(i_c, N)C} \tag{97}$$

Thus, Fig. 20, p.27:

$$S(N) = \frac{\delta_f(i_b, N)B}{\textit{BCR}_{\rm ref}} - \delta_f(i_c, N)C \tag{98}$$

Better (larger) *BCR*$_{\rm ref}$ requires better (lower) $S(N)$.
Positive solution exists for any *BCR*$_{\rm ref}$ such that the RHS of eq.(98) is positive:

$$\textit{BCR}_{\rm ref} < \frac{\delta_f(i_b, N)B}{\delta_f(i_c, N)C} \tag{99}$$

Reducing i_b or increasing i_c enables larger BCR_{ref}:

Reducing i_b increases discounted future benefits (because $\delta_f(i_b, N)$ increases).

Increasing i_c decreases discounted future costs (because $\delta_f(i_c, N)$ decreases).

The discounted fair price, eq.(98), fig. 20, with $BCR_{\text{ref}} = 0.02$:

Rises at low N because $\delta_f(i_b)$ and $\delta_f(i_c)$ rise at nearly the same rate.

Falls at high N because $\delta_f(i_c)$ rises faster than $\delta_f(i_b)$.

- Compare eq.(94) with no discounting of future benefits, $i_b = 0$:
  - ∘ $\delta_f(i_b = 0) = N = 5$.
  - ∘ Thus:

$$BCR(i_b = 0) = \frac{5}{3.3522} BCR(i_b = 0.15) = 1.4916 \times BCR(i_b = 0.15) = 0.0268 \tag{100}$$

  - ∘ Thus one undiscounted sortie-benefit costs $1/BCR = 1/.0268 = \$37.25 < \$55.56$.
  - ∘ The undiscounted sortie-benefit costs less because C_{pw} is distributed over more benefit.

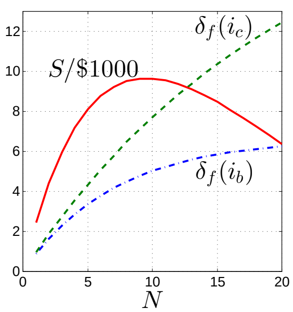

Figure 20: Discounted fair price and discount factors vs N. $BCR_{\text{ref}} = 0.02$.

3.5 Uncertain Discounting of Future Non-Monetary Benefit: Sorties of a Drone

- Continue section 3.4, p.25, and consider uncertain i_b and B (both constant over time):

$$\mathcal{U}(h) = \left\{ i_b, B: \ i_b > -1, \ \left| \frac{i_b - \widetilde{i}_b}{s_i} \right| \le h, \ \left| \frac{B - \widetilde{B}}{s_B} \right| \le h \right\}, \quad h \ge 0 \tag{101}$$

Questions: How to interpret s_i and s_B? How to formulation IGM if that information is lacking?

- Require:

$$\mathit{BCR}(i_b, B) \ge BCR_{\rm c} \tag{102}$$

for $\mathit{BCR}(i_b, B)$ from eq.(92), p.25.

- Robustness:

$$\widehat{h}(BCR_{\rm c}) = \max \left\{ h: \ \left(\min_{i_b, B \in \mathcal{U}(h)} \mathit{BCR}(i_b, B) \right) \ge \mathit{BCR}_c \right\} \tag{103}$$

- Inner minimum, $m(h)$, occurs at $i_b = \widetilde{i}_b + s_i h$ and $B = \widetilde{B} - s_b h$:

$$m(h) = \frac{\frac{1-(1+\widetilde{i}_b+s_i h)^{-N}}{\widetilde{i}_b+s_i h}(\widetilde{B} - s_B h)}{S + \frac{1-(1+i_c)^{-N}}{i_c} C} \tag{104}$$

Question: How to understand the "+" in $i_b = \widetilde{i}_b + s_i h$ and the "−" in $B = \widetilde{B} - s_b h$?
Why do they differ?

- Robustness curve in fig. 21, p.28.
  - ◦ Zeroing: $\widehat{h}(BCR_{\rm c}) = 0$ at $BCR_{\rm c} = 0.018 = \mathit{BCR}(\widetilde{i}_b, \widetilde{B})$, eq.(94), p.26.
  - ◦ Trade off: robustness rises as $BCR_{\rm c}$ falls.
    - — $\widehat{h}(BCR_{\rm c} = 0.01) = 2$. Reasonable or moderate robustness (**Why? When not?**).
    - — *BCR* $= 0.01$ implies $1/.01 = \$100$/sortie.
    - — Compare nominal, eq.(94), p.26: $1/0.018 = \$55.56$/sortie.
    - — Is \$55.56/sortie a fair or realistic price?
      \$55.56/sortie $\equiv$ 0.0180 sorties/\$ for which $\widehat{h} = 0$. Unreliable. Due to zeroing.

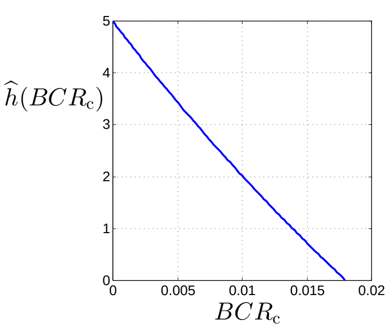

Figure 21: Robustness vs $BCR_{\rm c}$, eq.(104).

3.6 Probabilistic Uncertainty of Non-Monetary Benefit: Sorties of a Drone

- Continue section 3.4 with random benefit, B in eq.(93), p.25, $B \sim \mathcal{N}(\mu, \sigma^2)$.
  Question: How might we know that this is the pdf?
  - Theory: central limit theorem: sum of many iid events. (Not too plausible.)
  - Past experience, and assuming the future is similar. (Sometimes plausible.)
  - We focus on deep uncertainty, so pdf's typically unavailable or uncertain.
- The *BCR*, eqs.(92) and (93) p.25, is:

$$\textit{BCR} = \frac{\frac{1-(1+i_b)^{-N}}{i_b} B}{S + \frac{1-(1+i_c)^{-N}}{i_c} C} \tag{105}$$

$$= \underbrace{\frac{\delta_f(i_b)}{S + \delta_f(i_c) C}}_{Q} B, \quad \delta_f(i) \text{ defined in eq.(26), p.8} \tag{106}$$

- The probability of failure is:

$$P_{\rm f} = \text{Prob}(\textit{BCR} \le BCR_{\rm c}) = \text{Prob}(QB \le BCR_{\rm c}) = \text{Prob}\left(B \le \frac{BCR_{\rm c}}{Q}\right) \tag{107}$$

$$= \text{Prob}\left(\underbrace{\frac{B-\mu}{\sigma}}_{z\sim\mathcal{N}(0,1)} \le \frac{\frac{BCR_{\rm c}}{Q} - \mu}{\sigma} \right) \tag{108}$$

$$= \Phi\left(\frac{BCR_{\rm c} - Q\mu}{Q\sigma}\right) \tag{109}$$

- Note that, because $B \sim \mathcal{N}(\mu, \sigma^2)$ and $BCR = QB$:

$$\textit{BCR} \sim \mathcal{N}(Q\mu, Q^2\sigma^2) \tag{110}$$

Thus, when evaluating the probability of failure, we are usually interested in the case:

$$BCR_{\rm c} < Q\mu \tag{111}$$

Hence, assuming eq.(111):

$$\frac{\partial P_{\rm f}}{\partial \mu} \le 0 \quad \text{because } \frac{BCR_{\rm c} - Q\mu}{Q\sigma} \text{ gets **more** negative as } \mu \text{ increases} \tag{112}$$

$$\frac{\partial P_{\rm f}}{\partial \sigma} \ge 0 \quad \text{because } \frac{BCR_{\rm c} - Q\mu}{Q\sigma} \text{ gets **less** negative as } \sigma \text{ increases} \tag{113}$$

Eq.(112): Increased mean benefit, μ, causes reduced $P_{\rm f}$, fig. 22, left.
Eq.(113): Increased variance of benefit, σ^2, causes increased $P_{\rm f}$, fig. 22, right.

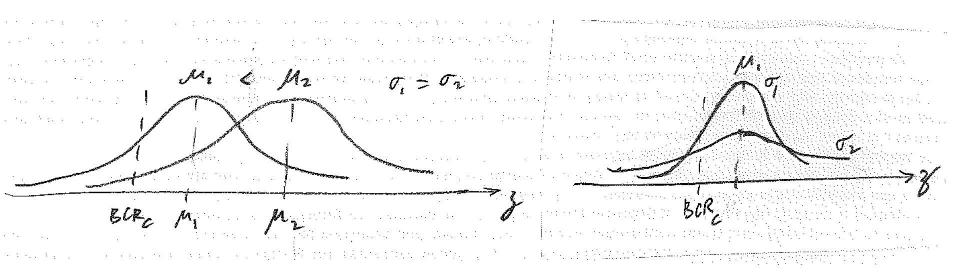

Figure 22: Probability distributions for various means and variances.

- Eq.(109) can be re-written:

$$P_{\rm f} = \Phi\left(\frac{BCR_{\rm c}}{Q\sigma} - \frac{\mu}{\sigma}\right) \tag{114}$$

Hence:

$$\frac{\partial P_{\rm f}}{\partial i_b} \geq 0 \quad \text{because } \delta_f(i_b)\downarrow \text{as } i_b\uparrow \text{ so } Q\downarrow \text{ so } \frac{BCR_{\rm c}}{Q\sigma} - \frac{\mu}{\sigma} \text{ gets \textbf{less} negative} \tag{115}$$

$$\frac{\partial P_{\rm f}}{\partial i_c} \leq 0 \quad \text{because } \delta_f(i_c)\downarrow \text{as } i_c\uparrow \text{ so } Q\uparrow \text{ so } \frac{BCR_{\rm c}}{Q\sigma} - \frac{\mu}{\sigma} \text{ gets \textbf{more} negative} \tag{116}$$

Eq.(115): increased discounting of benefits causes increased $P_{\rm f}$ by decreasing net benefit.
Eq.(116): increased discounting of cost causes decreased $P_{\rm f}$ by decreasing net cost.

3.7 Info-Gap Uncertain PDF of Non-Monetary Benefit: Sorties of a Drone

- Continue section 3.6, p.29, but with uncertain $p(B)$.
- Nominal estimate: $\widetilde{p}(B) \sim \mathcal{N}(\mu, \sigma^2)$. Fractional-error info-gap model for functional uncertainty:

$$\mathcal{U}(h) = \left\{ p(B) :\ p(B) \geq 0,\ \int_{-\infty}^{\infty} p(B)\,\mathrm{d}B = 1,\ \left| \frac{p(B) - \widetilde{p}(B)}{\widetilde{p}(B)} \right| \leq h \right\}, \quad h \geq 0 \tag{117}$$

- Note: eq.(117) is a modest info-gap model because uncertainty decays strongly on the tails.
- An info-gap model with greater uncertainty is:

$$\mathcal{U}(h) = \left\{ p(B) :\ p(B) \geq 0,\ \int_{-\infty}^{\infty} p(B)\,\mathrm{d}B = 1,\ \left| \frac{p(B) - \widetilde{p}(B)}{w} \right| \leq h \right\}, \quad h \geq 0 \tag{118}$$

  $w =$ constant, e.g. $w = \max_B \widetilde{p}(B)$. Large uncertainty on the tails.
- Probability of failure, from eq.(107), p.29:

$$P_{\mathrm{f}}(p) = \int_{-\infty}^{BCR_{\mathrm{c}}/Q} p(B)\,\mathrm{d}B \tag{119}$$

- Performance requirement:

$$P_{\mathrm{f}}(p) \leq P_{\mathrm{c}} \tag{120}$$

- Robustness:

$$\widehat{h}(P_{\mathrm{c}}) = \max \left\{ h :\ \left(\max_{p \in \mathcal{U}(h)} P_{\mathrm{f}}(p) \right) \leq P_{\mathrm{c}} \right\} \tag{121}$$

- Simplifying assumption (to make normalization easy), fig. 23:

$$BCR_{\mathrm{c}} \ll Q\mu \tag{122}$$

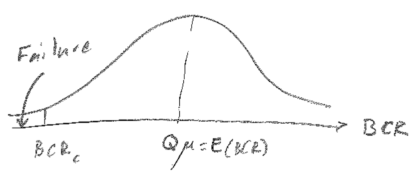

Figure 23: Eq.(122) implies low failure probability.

- Now the inner max in eq.(121), denoted $m(h)$, occurs at $p(B) = (1+h)\widetilde{p}(B)$ for $B \leq \frac{BCR_{\mathrm{c}}}{Q}$:

$$m(h) = (1+h) \int_{-\infty}^{BCR_{\mathrm{c}}/Q} \widetilde{p}(B)\,\mathrm{d}B = (1+h) P_{\mathrm{f}}(\widetilde{p}) \tag{123}$$

- Equate this to P_{c} and solve for h:

$$(1+h) P_{\mathrm{f}}(\widetilde{p}) = P_{\mathrm{c}} \quad \Longrightarrow \quad \widehat{h}(P_{\mathrm{c}}) = \frac{P_{\mathrm{c}}}{P_{\mathrm{f}}(\widetilde{p})} - 1 \tag{124}$$

  - Zeroing: $\widehat{h}(P_{\mathrm{c}}) = 0$ at $P_{\mathrm{c}} = P_{\mathrm{f}}(\widetilde{p})$.
  - Trade off: robustness increases as P_{c} increases.
- Robustness variation: analog to variation of P_{f}.
  - From eqs.(112), (113), p.29, and eq.(124):

$$\frac{\partial \widehat{h}}{\partial \mu} \geq 0 \tag{125}$$

$$\frac{\partial \widehat{h}}{\partial \sigma} \leq 0 \tag{126}$$

Eq.(125): Increased estimated mean benefit, μ, causes increased robustness, $\widehat{h}$.
Eq.(126): Increased estimated variance of benefit, σ^2, causes decreased robustness, $\widehat{h}$.

◦ From eqs.(115), (116), p.30, and eq.(124):

$$\frac{\partial \widehat{h}}{\partial i_b} \leq 0 \quad (127)$$

$$\frac{\partial \widehat{h}}{\partial i_c} \geq 0 \quad (128)$$

Eq.(125): Increased discounting of benefits, i_b, causes decreased robustness, $\widehat{h}$.
Eq.(126): Increased discounting of costs, i_c, causes increased robustness, $\widehat{h}$.

- Compare eqs.(112) and (113) with eqs.(125) and (126):

$$\frac{\partial P_{\rm f}}{\partial \mu} \leq 0, \ \frac{\partial P_{\rm f}}{\partial \sigma} \geq 0, \quad \frac{\partial \widehat{h}}{\partial \mu} \geq 0, \ \frac{\partial \widehat{h}}{\partial \sigma} \leq 0 \quad (129)$$

◦ $P_{\rm f}$ and $\widehat{h}$ respond in the same ways to change in μ or σ.
◦ Suggests that robustness could be a proxy for probability.[6]

- Compare eqs.(115) and (116) with eqs.(127) and (128):

$$\frac{\partial P_{\rm f}}{\partial i_b} \geq 0, \ \frac{\partial P_{\rm f}}{\partial i_c} \leq 0, \quad \frac{\partial \widehat{h}}{\partial i_b} \leq 0, \ \frac{\partial \widehat{h}}{\partial i_c} \geq 0 \quad (130)$$

◦ $P_{\rm f}$ and $\widehat{h}$ respond in the same ways to change in i_b or i_c.
◦ Suggests that robustness could be a proxy for probability.

[6]Yakov Ben-Haim, 2011, When is non-probabilistic robustness a good probabilistic bet? Working paper.
Yakov Ben-Haim, 2014, Robust satisficing and the probability of survival, *Intl. J. of System Science,* 45: 3-19.
Links to pre-prints of both articles here: https://info-gap.technion.ac.il/engineering-analysis-and-design/