41. **Present worth of yearly profit.** (Based on exam 28.5.2018.) (p.116)

(a) The profit at the end of year n is R_n, where R_1 = \$2,500, R_2 = \$3,500, R_3 = \$5,000. The discount rate is $i = 0.09$. What is the present worth of the total income stream?

(b) The profit at the end of year n is R_n, where R_1 = \$2,500, R_2 = \$3,500, R_3 = \$5,000. At the end of each year you will invest that year's profit, R_n, with yearly rate of return of $i_{\mathrm{a}} = 0.15$. What is the total accumulated sum at the end of year 3? What is the present worth of that sum using a discount rate of $i = 0.09$?

(c) The profit at the end of year n is R_n, where the estimated values of these profits are $\widetilde{R}_n$ for $n = 1, 2, 3$. The uncertainty in these estimates is given by this info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R_n - \widetilde{R}_n}{\widetilde{R}_n} \right| \le h, \ \ n = 1, 2, 3 \right\}, \quad h \ge 0 \tag{36}$$

The discount rate is i. You require that the present worth be no less than the critical value $\mathrm{PW_c}$. Derive an explicit algebraic expression for the robustness.

(d) The return on an investment is a random variable, R, in the interval $[R_1, \ R_2]$. The investment is a success if the return exceeds the critical value R_{c}. The probability of success is:

$$P_{\mathrm{s}}(R_{\mathrm{c}}) = \begin{cases} 0 & \text{if } R_{\mathrm{c}} > R_2 \\ \dfrac{R_2 - R_{\mathrm{c}}}{R_2 - R_1}, & \text{if } R_1 \le R_{\mathrm{c}} \le R_2 \\ 1 & R_{\mathrm{c}} < R_1 \end{cases} \tag{37}$$

However, the value of the critical return, R_{c}, is uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ R_{\mathrm{c}} : \left| \frac{R_{\mathrm{c}} - \widetilde{R}_{\mathrm{c}}}{\widetilde{R}_{\mathrm{c}}} \right| \le h \right\}, \quad h \ge 0 \tag{38}$$

You require that the probability of success be no less than the critical value P_{c}. Derive an explicit algebraic expression for the robustness. Assume that $R_1 \le \widetilde{R}_{\mathrm{c}} \le R_2$.

(e) (Variation on part 41a) The profit at the end of year n is R_n, where R_1 = \$5000, R_2 = \$3,500, R_3 = \$2,500. The discount rate is $i = 0.05$. What is the present worth of the total income stream?

(f) (Variation of part 41b) The profit at the end of year n is R_n, where R_1 = \$5,000, R_2 = \$3,500, R_3 = \$2,500. At the end of each year you will invest that year's profit, R_n, with yearly rate of return of $i_{\mathrm{a}} = 0.1$. What is the total accumulated sum at the end of year 3? What is the present worth of that sum using a discount rate of $i = 0.04$?

(g) (Variation on part 41c) The profit at the end of year n is R_n, where the estimated values of these profits are $\widetilde{R}_n$ for $n = 1, 2, 3$. The uncertainty in these estimates is given by this info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R_n - \widetilde{R}_n}{w} \right| \le h, \ \ n = 1, 2, 3 \right\}, \quad h \ge 0 \tag{39}$$

where w is a known positive constant. The discount rate is i. You require that the present worth be no less than the critical value $\mathrm{PW_c}$. Derive an explicit algebraic expression for the robustness.

(h) (Variation of part 41d) The return on an investment is a random variable, R, in the interval $[R_1,\ R_2]$. The investment is a success if the return exceeds the critical value $R_{\rm c}$. The probability of success is:

$$P_{\rm s}(R_{\rm c}) = \begin{cases} 0 & \text{if } R_{\rm c} > R_2 \\ \dfrac{R_2 - R_{\rm c}}{R_2 - R_1}, & \text{if } R_1 \le R_{\rm c} \le R_2 \\ 1 & R_{\rm c} < R_1 \end{cases} \tag{40}$$

However, the value of the critical return, $R_{\rm c}$, is uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ R_{\rm c} : \ \left| \frac{R_{\rm c} - \widetilde{R}_{\rm c}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{41}$$

where w is a known positive constant. You require that the probability of success be no less than the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness. Assume that $R_1 \le \widetilde{R}_{\rm c} \le R_2$.

Solution to problem 41, **Present worth of yearly profit** (p.38).

(41a) The present worth is:

$$\text{PW} = \sum_{n=1}^{N}(1+i)^{-n}R_n = 1.09^{-1}2,500 + 1.09^{-2}3,500 + 1.09^{-3}5,000 = \$9,100.4 \tag{534}$$

(41b) The total accumulated value at the end of year 3 is:

$$V = \sum_{n=1}^{N}(1+i_{\rm a})^{N-n}R_n = 1.15^{2}2,500 + 1.15^{1}3,500 + 1.15^{0}5,000 = \$12,331 \tag{535}$$

The present worth of this value is:

$$\text{PW}(V) = (1+i)^{-3}V = 1.09^{-3}12,331 = \$9,521.8 \tag{536}$$

(41c) The present worth of the profit stream is:

$$\text{PW}(R) = \sum_{n=1}^{N}(1+i)^{-n}R_n \tag{537}$$

The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R\in\mathcal{U}(h)} \text{PW}(R)\right) \geq \text{PW}_{\rm c}\right\} \tag{538}$$

Let $m(h)$ denote the inner minimum, which occurs when each profit is as low as possible at horizon of uncertainty h:

$$m(h) = \sum_{n=1}^{N}(1+i)^{-n}(1-h)\widetilde{R}_n = (1-h)\text{PW}(\widetilde{R}) \tag{539}$$

Equating this to the critical value, $\text{PW}_{\rm c}$, and solving for h yields the robustness:

$$\widehat{h} = 1 - \frac{\text{PW}_{\rm c}}{\text{PW}(\widetilde{R})} \tag{540}$$

or zero if this is negative.

(41d) The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R_{\rm c}\in\mathcal{U}(h)} P_{\rm s}(R_{\rm c})\right) \geq P_{\rm c}\right\} \tag{541}$$

Let $m(h)$ denote the inner minimum, which occurs when the critical value, $R_{\rm c}$, is as large as possible at horizon of uncertainty h:

$$m(h) = \begin{cases} \dfrac{R_2 - (1+h)\widetilde{R}_{\rm c}}{R_2 - R_1} & \text{if } (1+h)\widetilde{R}_{\rm c} \leq R_2 \ \ (\text{equiv: } h \leq \dfrac{R_2}{\widetilde{R}_{\rm c}} - 1) \\ 0 & \text{else} \end{cases} \tag{542}$$

Equating this to the critical value, $P_{\rm c}$, and solving for h yields the robustness:

$$\widehat{h} = \frac{R_2 - (R_2 - R_1)P_{\rm c}}{\widetilde{R}_{\rm c}} - 1 \tag{543}$$

or zero if this is negative. Note that $\widehat{h} \leq \dfrac{R_2}{\widetilde{R}_{\rm c}} - 1$.

(41e) The present worth is:

$$\text{PW} = \sum_{n=1}^{N} (1+i)^{-n} R_n = 1.05^{-1} 5,000 + 1.05^{-2} 3,500 + 1.05^{-3} 2,500 = \$10,096 \tag{544}$$

(41f) The total accumulated value at the end of year 3 is:

$$V = \sum_{n=1}^{N} (1+i_{\text{a}})^{N-n} R_n = 1.1^2 5,000 + 1.1^1 3,500 + 1.1^0 2,500 = \$12,400 \tag{545}$$

The present worth of this value is:

$$\text{PW}(V) = (1+i)^{-3} V = 1.04^{-3} 12,400 = \$11,024 \tag{546}$$

(41g) The present worth of the profit stream is:

$$\text{PW}(R) = \sum_{n=1}^{N} (1+i)^{-n} R_n \tag{547}$$

The definition of the robustness is:

$$\widehat{h} = \max \left\{ h : \ \left(\min_{R \in \mathcal{U}(h)} \text{PW}(R) \right) \geq \text{PW}_{\text{c}} \right\} \tag{548}$$

Let $m(h)$ denote the inner minimum, which occurs when each profit is as low as possible at horizon of uncertainty h:

$$m(h) = \sum_{n=1}^{N} (1+i)^{-n} \left(\widetilde{R}_n - wh \right) = \text{PW}(\widetilde{R}) - h\text{PW}(w) \tag{549}$$

Equating this to the critical value, PW_{c}, and solving for h yields the robustness:

$$\widehat{h} = \frac{\text{PW}(\widetilde{R}) - \text{PW}_{\text{c}}}{\text{PW}(w)} \tag{550}$$

or zero if this is negative.

(41h) The definition of the robustness is:

$$\widehat{h} = \max \left\{ h : \ \left(\min_{R_{\text{c}} \in \mathcal{U}(h)} P_{\text{s}}(R_{\text{c}}) \right) \geq P_{\text{c}} \right\} \tag{551}$$

Let $m(h)$ denote the inner minimum, which occurs when the critical value, R_{c}, is as large as possible at horizon of uncertainty h:

$$m(h) = \left\{ \begin{array}{cl} \dfrac{R_2 - (\widetilde{R}_{\text{c}} + wh)}{R_2 - R_1} & \text{if } \widetilde{R}_{\text{c}} + wh \leq R_2 \ \ (\text{equiv: } h \leq \dfrac{R_2 - \widetilde{R}_{\text{c}}}{w}) \\ 0 & \text{else} \end{array} \right. \tag{552}$$

Equating this to the critical value, P_{c}, and solving for h yields the robustness:

$$\widehat{h} = \frac{R_2 - \widetilde{R}_{\text{c}} - (R_2 - R_1) P_{\text{c}}}{w} \tag{553}$$

or zero if this is negative. Note that $\widehat{h} \leq \dfrac{R_2 - \widetilde{R}_{\text{c}}}{w}$.