54. **Loans and More.** (Based on midterm exam, 16.7.2024). (p.143). The 4 parts of this problem are independent of one another.

(a) You will take a loan of $\$L$ at time $t = 0$. You will return the loan in 3 payments, R_1, R_2 and R_3 at the end of years 1, 2 and 3, respectively, where $L = R_1 + R_2 + R_3$. At each loan payment you will also pay the accumulated interest at annual interest rate i. These interest payments are denoted I_1, I_2 and I_3. State an explicit algebraic expression for each of these three interest payments.

(b) You will earn R_1, R_2 and R_3 dollars at the end of years 1, 2 and 3, respectively. Each sum is immediately invested with annual rate of return i. What is the total value, $\$T$, of these investments at the end of year 3? What is the present worth (at time $t = 0$) of $\$T$, with discount rate i?

(c) The present worth of a particular project is:

$$\text{PW} = bR \tag{94}$$

where b is a known positive constant and R is uncertain as specified by this info-gap model of uncertainty:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R - \widetilde{R}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{95}$$

where w and $\widetilde{R}$ are known positive constants. We require that the present worth be no less than the critical value PW_c. Derive an explicit algebraic expression for the robustness to uncertainty for achieving this requirement.

(d) Two different economic projects are evaluated by their future worths, which are uncertain. The robustness functions for the future worths of these projects are:

$$\widehat{h}_1(\text{FW}_\text{c}) = \frac{a - \text{FW}_\text{c}}{b} \tag{96}$$

$$\widehat{h}_2(\text{FW}_\text{c}) = \frac{c - \text{FW}_\text{c}}{d} \tag{97}$$

or zero if the corresponding function is negative, where:

$$0 < c < a \quad \text{and} \quad 0 < d < b \tag{98}$$

Derive an explicit algebraic expression for the range of values of FW_c for which project 1 is preferred over project 2 according to the robustness criterion.

Solution to problem 54 **Loans and More.** (p.52).

54a. At time $t = 0$ you take a loan of L. After 1 year the accumulated interest, at rate i, is:

$$I_1 = iL \tag{749}$$

You now return the amount R_1 and the interest I_1, so you now hold a loan of $L - R_1$. At the end of the 2nd year you have held this loan for 1 year, so the accumulated interest, at rate i, is:

$$I_2 = i(L - R_1) \tag{750}$$

You now return the amount R_2 and the interest I_2, so you now hold a loan of $L - R_1 - R_2$. At the end of the 3rd year you have held this loan for 1 year, so the accumulated interest, at rate i, is:

$$I_3 = i(L - R_1 - R_2) \tag{751}$$

In summary, the 3 interest payments are:

$$I_1 = iL, \quad I_2 = i(L - R_1), \quad I_1 = i(L - R_1 - R_2) \tag{752}$$

54b. The future worth of these investments (total value at the end of year 3) is:

$$T = \text{FW} = (1+i)^2 R_1 + (1+i)R_2 + R_3 \tag{753}$$

The present worth (at the same rate of return) is:

$$\text{PW} = (1+i)^{-3}\left[(1+i)^2 R_1 + (1+i)R_2 + R_3\right] \tag{754}$$

$$= (1+i)^{-1} R_1 + (1+i)^{-2} R_2 + (1+i)^{-3} R_3 \tag{755}$$

54c. The definition of the robustness function is:

$$\widehat{h}(\text{PW}_\text{c}) = \max\left\{h : \left(\min_{R\in\mathcal{U}(h)} \text{PW}\right) \geq \text{PW}_\text{c}\right\} \tag{756}$$

Let $m(h)$ denote the inner minimum, which occurs when $R = \widetilde{R} - wh$. Thus:

$$m(h) = b(\widetilde{R} - wh) \geq \text{PW}_\text{c} \quad \Longrightarrow \quad \widehat{h}(\text{PW}_\text{c}) = \frac{b\widetilde{R} - \text{PW}_\text{c}}{bw} \tag{757}$$

or zero if this is negative.

54d. The robustness curves are shown schematically in fig. 18, p.144.

Equate the two robustness functions to find the value of FW_c at which the robustness curves intersect:

$$\widehat{h}_1(\text{FW}_\times) = \widehat{h}_2(\text{FW}_\times) \iff \frac{a - \text{FW}_\times}{b} = \frac{c - \text{FW}_\times}{d} \iff ad - d\text{FW}_\times = cb - b\text{FW}_\times \tag{758}$$

Hence:

$$(d - b)\text{FW}_\times = ad - cb \iff \text{FW}_\times = \frac{ad - cb}{d - b} \tag{759}$$

Thus project 1 is robust-preferred for:

$$\text{FW}_\times < \text{FW} < a \tag{760}$$

Note the strict inequalities.

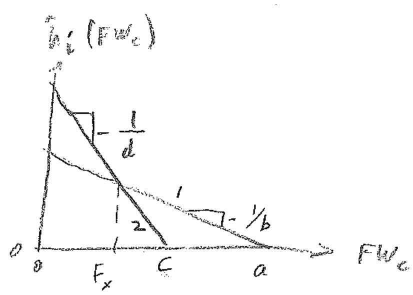

Figure 18: Schematic robustness curves for solution of problem 54d.