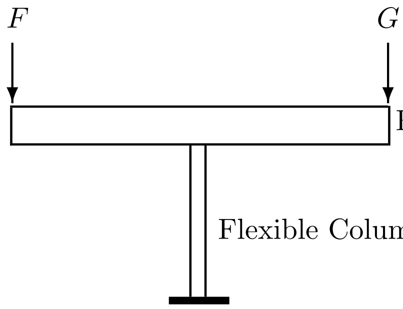

Figure 1: Platform for problem 9.

9. **Dynamic stability of a platform.** (p.50) A rigid beam-like platform is supported from below at its midpoint by a flexible column which is at elastic equilibrium when the platform is horizontal, as shown in fig. 1. The flexural stiffness of the elastic column is k [Nm/radian] and it applies a restoring moment of force $M = k\theta$ when the platform is tilted by θ radians. The width of the platform is $2L$ [m]. The platform is loaded at its two ends by static forces F and G which are uncertain but bounded. That is, forces F and G belong to the following info-gap model of uncertainty:

$$\mathcal{U}(h, 0) = \{F, G: \ |F| \ \le \ h, \quad |G| \ \le \ h\}, \quad h \ge 0 \tag{24}$$

The platform is satisfactorily level if the angle of tilt at static equilibrium is never greater than the critical value θ_c:

$$|\theta| \le \theta_c \tag{25}$$

The condition of static equilibrium requires that the moment of force at the midpoint vanish:

$$0 = FL - GL + k\theta \tag{26}$$

Determine the robustness and opportuneness functions of the platform. The decision vector is $q = (k, L)^T$. Study the variation of the immunity functions as these design variables are changed.

10. **Dynamic stability of a platform: continued.** (p.51) We now modify problem 9 to consider uncertain distributed loads, $f(x)$ [N/m], $-L \le x \le L$, on the platform. Evaluate the robustness and opportuneness for each of the following info-gap models for uncertainty in the load.

    (a) *Uniform-bound:*

    $$\mathcal{U}(h, \widetilde{f}) = \left\{ f(x): \ \left| f(x) - \widetilde{f} \cos \frac{\pi x}{L} \right| \le h \right\}, \quad h \ge 0 \tag{27}$$

    where $\widetilde{f}$ is a known constant.

    (b) *Fourier ellipsoid bound:* The uncertain part of the load profile is a truncated sine series:

    $$f(x) = \widetilde{f} \cos \frac{\pi x}{L} + \sum_{n=1}^{N} c_n \sin \frac{n\pi x}{L} \tag{28}$$

    $$= \widetilde{f} \cos \frac{\pi x}{L} + c^T \sigma(x) \tag{29}$$

    where c is the vector of uncertain Fourier coefficients and $\sigma(x)$ is the vector of sine functions. The info-gap model is:

    $$\mathcal{U}(h, \widetilde{f}) = \left\{ f(x) = \widetilde{f} \cos \frac{\pi x}{L} + c^T \sigma(x): \ c^T W c \le h^2 \right\}, \quad h \ge 0 \tag{30}$$

    where W is a known, real, symmetric, positive definite matrix.

(c) *Different nominal load.* How will the answers to questions 10a and 10b change if the nominal load is:

$$\widetilde{f}(x) = \widetilde{f} \sin \frac{\pi x}{L} \tag{31}$$

Solution to Problem 9. (p.8) **Solution: Static equilibrium.** The mechanical model is the condition for static equilibrium. Since the platform is rigid, equilibrium requires that the moment at the midpoint is zero:

$$0 = FL - GL + k\theta \tag{308}$$

which implies:

$$\theta = \frac{(G-F)L}{k} \tag{309}$$

The robustness is the greatest value of the uncertainty parameter h such that failure cannot occur:

$$\widehat{h} = \max \left\{ h : \max_{F,\, G \in \mathcal{U}(h)} |\theta| \le \theta_{\rm c} \right\} \tag{310}$$

The maximum θ up to uncertainty h is:

$$\max_{F,G \in \mathcal{U}(h)} \theta = \frac{(h-(-h))L}{k} = \frac{2hL}{k} \tag{311}$$

The robust reliability is obtained by equating the maximum deflection to the critical value and solving for h:

$$\max_{F,G \in \mathcal{U}(h)} \theta = \theta_{\rm c} \quad \Longrightarrow \quad \frac{2hL}{k} = \theta_{\rm c} \quad \Longrightarrow \quad \widehat{h} = \frac{\theta_{\rm c} k}{2L} \tag{312}$$

Solution: Rotational vibration. Let μ [kg/m] be the linear mass density of the platform in the horizontal direction. The moment of inertia is:

$$J = 2 \int_0^L x^2 \mu \, {\rm d}x = \frac{2}{3} \mu L^3 \tag{313}$$

The equation of rotational vibration around the midpoint is:

$$J\ddot{\theta} + k\theta = M, \quad \theta(0) = \dot{\theta}(0) = 0 \tag{314}$$

where M is the external moment of force at the midpoint:

$$M = (F - G)L \tag{315}$$

The solution of eq.(328) is:

$$\theta(t) = \frac{1}{J\omega} \int_0^t M \sin \omega\tau \, {\rm d}\tau \tag{316}$$

$$= \frac{M}{J\omega^2}(1 - \cos\omega t) \tag{317}$$

where the natural frequency is $\omega = \sqrt{k/J}$.

The robustness is the greatest value of the uncertainty parameter h such that failure cannot occur:

$$\widehat{h} = \max \left\{ h : \max_{F,\, G \in \mathcal{U}(h)} |\theta| \le \theta_{\rm c} \right\} \tag{318}$$

The maximum in this expression is:

$$\max_{F,\, G \in \mathcal{U}(h)} |\theta| = \frac{(h-(-h))L}{J\omega^2}(1 - \cos\omega t) = \frac{2hL}{J\omega^2}(1 - \cos\omega t) \tag{319}$$

Equating this maximum to the critical angle and solving for the uncertainty parameter yields the robustness:

$$\frac{2hL}{J\omega^2}(1 - \cos\omega t) = \theta_{\rm c} \quad \Longrightarrow \quad \widehat{h}(t) = \frac{J\omega^2 \theta_{\rm c}}{2L(1 - \cos\omega t)} \tag{320}$$

Note that $\widehat{h}(t)$ is not monotonic versus t, but periodically approaches infinity.

In some situations we will be interested in the minimum over time of $\widehat{h}(t)$:

$$\widehat{h} = \min_t \widehat{h}(t) = \frac{J\omega^2\theta_c}{4L} \tag{321}$$

Solution to Problem 10. (p.8) **(a, part 1) Uniform-bound uncertainty; static equilibrium.** A condition for static mechanical equilibrium is balance of the torque at the midpoint:

$$0 = k\theta - \int_{-L}^{L} xf(x)\,\mathrm{d}x \tag{322}$$

Therefore we adopt the following mechanical model:

$$\theta = \frac{1}{k}\int_{-L}^{L} xf(x)\,\mathrm{d}x \tag{323}$$

The robustness is the greatest uncertainty which does not entail the possibility of failure:

$$\widehat{h} = \max\left\{h:\ \max_{f\in\mathcal{U}(h,\widetilde{f})} |\theta| \le \theta_c\right\} \tag{324}$$

As before, we must find the maximum deflection up to uncertainty h. This maximum occurs when the load $f(x)$ is minimum ($f(x) = -h + \widetilde{f}\cos\frac{\pi x}{L}$) when x is negative, and maximal ($f(x) = +h + \widetilde{f}\cos\frac{\pi x}{L}$) when x is positive:

$$\max_{f\in\mathcal{U}(h,\widetilde{f})} \theta = \frac{1}{k}\int_{-L}^{0}\left[-h + \widetilde{f}\cos\frac{\pi x}{L}\right]x\,\mathrm{d}x + \frac{1}{k}\int_{0}^{L}\left[h + \widetilde{f}\cos\frac{\pi x}{L}\right]x\,\mathrm{d}x \tag{325}$$

The two terms containing $\widetilde{f}$ cancel each other, resulting in:

$$\max_{f\in\mathcal{U}(h,\widetilde{f})} \theta = \frac{2h}{k}\int_{0}^{L} x\,\mathrm{d}x = \frac{hL^2}{k} \tag{326}$$

Equating this maximum deflection to the critical value θ_c and solving for the uncertainty parameter h, yields the robustness:

$$\frac{hL^2}{k} = \theta_c \quad\Longrightarrow\quad \widehat{h} = \frac{\theta_c k}{L^2} \tag{327}$$

(a, part 2) Uniform-bound uncertainty; rotational vibration. We could also analyze this problem dynamically rather than statically. That is, we consider the motion of the platform from zero initial conditions, with constant but uncertain load. The equation of motion is:

$$J\ddot{\theta} + k\theta = M, \quad \theta(0) = \dot{\theta}(0) = 0 \tag{328}$$

where the moment of inertial J is:

$$J = 2\int_{0}^{L} x^2\mu\,\mathrm{d}x = \frac{2}{3}\mu L^3 \tag{329}$$

The torque at the midpoint is:

$$M = -\int_{-L}^{L} xf(x)\,\mathrm{d}x \tag{330}$$

We can think of this load as a linear superposition of inputs: $xf(x)$ at each x. Thus the output is the linear superposition of responses:

$$\theta(t) = -\frac{1}{J\omega}\int_{-L}^{L}\int_{0}^{t} xf(x)\sin\omega\tau\,\mathrm{d}\tau\,\mathrm{d}x \tag{331}$$

These two integrals can be separated, resulting in:

$$\theta(t) = \frac{1}{J\omega^2}(1 - \cos\omega t)\int_{-L}^{L} x f(x)\,\mathrm{d}x \tag{332}$$

Since the load is constant, the sign of the angle of deflection is constant, unlike for an impulse load.

The robustness is the greatest h which does not entail failure:

$$\widehat{h} = \max\left\{h:\ \max_{f\in\mathcal{U}(h,\widetilde{f})} |\theta| \le \theta_{\rm c}\right\} \tag{333}$$

Arguing as in eqs.(325) and (326), the maximum of the integral in eq.(332) becomes:

$$\max_{f\in\mathcal{U}(h,\widetilde{f})}\int_{-L}^{L} x f(x)\,\mathrm{d}x = hL^2 \tag{334}$$

Now equating the maximum deflection to the critical deflection and solving for h yields the robustness:

$$\frac{hL^2}{J\omega^2}(1-\cos\omega t) = \theta_{\rm c} \quad\Longrightarrow\quad \widehat{h}(t) = \frac{J\omega^2\theta_{\rm c}}{L^2(1-\cos\omega t)} \tag{335}$$

Note the periodic minima and poles of $\widehat{h}(t)$. The horizon of uncertainty, h, in the info-gap model of eq.(27), has units of [N/m], which are also the units of $\widehat{h}$.

(b, part 1) Fourier-ellipsoid bound uncertainty; static equilibrium. From eq.(323), the nominal deflection vanishes because $\widetilde{f}(x)$ is an even function:

$$\widetilde{\theta} = \frac{1}{k}\int_{-L}^{L} x\widetilde{f}(x)\,\mathrm{d}x = 0 \tag{336}$$

Hence, the total deflection is:

$$\begin{aligned}
\theta &= \frac{1}{k}\int_{-L}^{L} x f(x)\,\mathrm{d}x && (337)\\
&= \frac{1}{k}\int_{-L}^{L} x c^T\sigma(x)\,\mathrm{d}x && (338)\\
&= \frac{1}{k}\sum_{n=1}^{N} c_n \int_{-L}^{L} x\sin\frac{n\pi x}{L}\,\mathrm{d}x && (339)\\
&= \frac{2L^2}{k\pi}\sum_{n=1}^{N}(-1)^{n+1}\frac{c_n}{n} && (340)\\
&= c^T z && (341)
\end{aligned}$$

where we have defined an N-vector z whose elements are:

$$z_n = (-1)^{n+1}\frac{2L^2}{n\pi k} \tag{342}$$

The maximum deflection, up to uncertainty h, is the solution of the following optimization problem:

$$\max_{f\in\mathcal{U}(h)}\theta = \max_{c^TWc=h^2} c^T z \tag{343}$$

We will use the method of Lagrange multipliers to find this maximum. Define the auxiliary function:

$$H = c^T z + \lambda\left[h^2 - c^T W c\right] \tag{344}$$

λ is the unknown Lagrange multiplier. Its value is found by employing the constraint:

$$h^2 = c^T W c \tag{345}$$

The term in the square brackets in eq.(344) equals zero when the constraint is observed. Thus, maximizing H is equivalent to maximizing $c^T z$. Thus, we must solve:

$$0 = \frac{\partial H}{\partial c} \tag{346}$$

Differentiating H in eq.(344) results in:

$$\frac{\partial H}{\partial c} = z - 2\lambda W c \tag{347}$$

(Recall that W is a symmetric matrix.) Eq.(347) implies that an optimizing c-vector is:

$$c = \frac{1}{2\lambda} W^{-1} z \tag{348}$$

Substituting from eq.(348) for c into the constraint in eq.(345) one finds:

$$h^2 = \frac{1}{4\lambda^2} z^T W^{-1} W W^{-1} z \tag{349}$$

Thus the constraint determines the Lagrange multiplier as:

$$\frac{1}{2\lambda} = \frac{\pm h}{\sqrt{z^T W^{-1} z}} \tag{350}$$

Now, combining this with eq.(348), one finds the vector of Fourier coefficients which optimize the deflection to be:

$$c = \frac{\pm h}{\sqrt{z^T W^{-1} z}} W^{-1} z \tag{351}$$

Hence, the extremal angles of deflection are:

$$\max_{f \in \mathcal{U}(h)} \theta = \max_{c^T W c = h^2} c^T z = \pm h \sqrt{z^T W^{-1} z} \tag{352}$$

Equating the maximum absolute deflection to the critical deflection yields the robustness:

$$h \sqrt{z^T W^{-1} z} = \theta_{\mathrm{c}} \quad \Longrightarrow \quad \widehat{h} = \frac{\theta_{\mathrm{c}}}{\sqrt{z^T W^{-1} z}} \tag{353}$$

Solution: (b, part 2) Fourier-ellipsoid bound uncertainty; rotational vibration. The moment, which is constant, is:

$$M = -\int_{-L}^{L} x f(x)\,\mathrm{d}x = -\int_{-L}^{L} x \widetilde{f}(x)\,\mathrm{d}x - c^T \int_{-L}^{L} x \sigma(x)\,\mathrm{d}x = \widetilde{M} + c^T z \tag{354}$$

where $\widetilde{M}$ is the nominal, anticipated moment, and it and z are known and defined in this relation.

The angle of deflection is:

$$\theta(t) = \frac{1}{J\omega} \int_0^t M \sin \omega\tau\,\mathrm{d}\tau = (\widetilde{M} + c^T z) \frac{1}{J\omega} \int_0^t \sin \omega\tau\,\mathrm{d}\tau = (\widetilde{M} + c^T z) \frac{1}{J\omega^2} (1 - \cos \omega t) \tag{355}$$

The robustness is:

$$\widehat{h} = \max \left\{ h : \max_{f \in \mathcal{U}(h, \widetilde{f})} |\theta| \le \theta_{\mathrm{c}} \right\} \tag{356}$$

To evaluate the robustness we must maximize $c^T z$, for which we employ Lagrange optimization. Define:

$$H = c^T z + \lambda (h^2 - c^T W c) \tag{357}$$

The extrema occur at:

$$0 = \frac{\partial H}{\partial c} = z - 2\lambda W c \tag{358}$$

which implies that:

$$c = \frac{1}{2\lambda} W^{-1} z \tag{359}$$

Employing the constraint, $h^2 = c^T W c$, we find that the optimizing c is:

$$c = \frac{\pm h}{\sqrt{z^T W^{-1} z}} W^{-1} z \tag{360}$$

From this we obtain:

$$\max_{f \in \mathcal{U}(h, \widetilde{f})} |\theta| = \left(|\widetilde{M}| + h\sqrt{z^T W^{-1} z} \right) \frac{1}{J\omega^2} (1 - \cos \omega t) \tag{361}$$

The robustness is the greatest value of h at which this expression does not exceed $\theta_{\rm c}$:

$$\widehat{h} = \left(\frac{\theta_{\rm c}}{J\omega^2 (1 - \cos \omega t)} - |\widetilde{M}| \right) \frac{1}{\sqrt{z^T W^{-1} z}} \tag{362}$$

unless this expression is negative, in which case $\widehat{h} = 0$.