**Lecture Notes on**

# Time-Value of Money

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**Source material:**

• DeGarmo, E. Paul, William G. Sullivan, James A. Bontadelli and Elin M. Wicks, 1997, *Engineering Economy.* 10th ed., chapters 3–4, Prentice-Hall, Upper Saddle River, NJ.

• Ben-Haim, Yakov, 2010, *Info-Gap Economics: An Operational Introduction,* Palgrave-Macmillan.

• Ben-Haim, Yakov, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London.

**A Note to the Student:** These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

## Contents



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§ **The problem:**

- Given several different design concepts, each technologically acceptable.
- Select one option or prioritize all the options.

§ **The economic approach:**

- Treat each option as a *capital investment.*
- Consider:
  - *Associated expenditures* for implementation.
  - *Revenues* or *savings* over time.
  - Attractive or acceptable *return on investment.*
  - Cash flows over time: **time-value of money.**

§ **Why should the engineer study economics?**

- Cost and revenue are unavoidable in practical engineering in industry, government, etc.
- The engineer must be able to communicate and collaborate with the economist:
  - Economic decisions depend on engineering considerations.
  - Engineering decisions depend on economic considerations.
- Technology influences society, and society influences technology:
  Engineering is both a technical and a social science.[1]

§ We will deal with **design-prioritization** in part II, p.16.

§ We first study the **time-value of money** in part I on p.4.

§ In part III we will study the **implications of uncertainty.**

[1] Yakov Ben-Haim, 2000, Why the best engineers should study humanities, *Intl J for Mechanical Engineering Education,* 28: 195–200. Link to pre-print on: http://info-gap.com/content.php?id=23

# Part I
# Time-Value of Money

## 1 Time, Money and Engineering Design

§ **Design problem: discrete options.**

- Goal: design system for 10-year operation.
- Option 1: High quality, expensive 10-year components.
- Option 2: Medium quality, less expensive 5-year components. Re-purchase after 5 years.
- Which design preferable?
  - What are the considerations?
  - How to compare costs?

§ **Design problem: continuous options.**

- Goal: design system for 10-year operation.
- Many options, allowing continuous trade off between price and life.
- Which design preferable?
  - What are the considerations?
  - How to compare costs?

§ **Repair options.**

- The production system is broken.
- When functional, the system produces goods worth $500,000 per year.
- Various repair technologies have different costs and projected lifetimes.
- How much can we spend on repair that would return the system to $N$ years of production?
- Which repair technology should we use?
- Should we look for other repair technologies?

## 2 Simple Interest

§ **Primary source:** DeGarmo *et al,* p.65.

§ **Interest:** "Money paid for the use of money lent (the principal), or for forbearance of a debt, according to a fixed ratio".[2]

§ **Biblical prohibition:** "If you lend money to any of my people with you that is poor, you shall not be to him as a creditor; nor shall you lay upon him interest."[3] (transparency)

§ **Simple interest:**[4] The total amount of interest paid is *linearly proportional* to:
- Initial loan, $P$, (the principal).[5]
- The number of periods, $N$.

§ **Interest rate,** $i$:
- Proportionality constant.
- E.g., 10% interest: $i = 0.1$.

§ **Total interest payment,** $I$, on principal $P$ for $N$ periods at interest rate $i$:

$$I = PNi \tag{1}$$

Example: $P = \$200$, $N = 5$ periods (e.g. years), $i = 0.1$:

$$I = \$200 \times 5 \times 0.1 = \$100 \tag{2}$$

§ **Total repayment:**

$$C = (1 + Ni)P \tag{3}$$

§ We will **not use** simple interest because it is not used in practice.

[2] *OED,* online, 21.9.2012.
[3] *Exodus,* 22:24.
[4] Interest: rebeet. "rebeet" is written with taf.
[5] Principal: keren.

# 3 Compound Interest

§ **Primary source:** DeGarmo *et al,* p.66.

§ **Compound interest:**[6] The interest charge for any period is linearly proportional to both:
- Remaining principal, and
- Accumulated interest up to beginning of that period.

**Example 1** 4 different compound-interest schemes. See table 1
- $8,000 principal at 10% annually for 4 years.
- Plan 1: At end of each year pay $2,000 plus interest due.
- Plan 2: Pay interest due at end of each year, and pay principal at end of 4 years.
- Plan 3: Pay in 4 equal end-of-year payments.
- Plan 4: Pay principal and interest in one payment at end of 4 years.

■

| Year | Amount owed at beginning of year | Interest accrued for year | Principal payment | Total end-of-year payment |
|---|---|---|---|---|
| **Plan 1:** | | | | |
| 1 | 8,000 | 800 | 2,000 | 2,800 |
| 2 | 6,000 | 600 | 2,000 | 2,600 |
| 3 | 4,000 | 400 | 2,000 | 2,400 |
| 4 | 2,000 | 200 | 2,000 | 2,200 |
| Total: | 20,000 $-yr | 2,000 | 8,000 | 10,000 |
| **Plan 2:** | | | | |
| 1 | 8,000 | 800 | 0 | 800 |
| 2 | 8,000 | 800 | 0 | 800 |
| 3 | 8,000 | 800 | 0 | 800 |
| 4 | 8,000 | 800 | 8,000 | 8,800 |
| Total: | 32,000 $-yr | 3,200 | 8,000 | 11,200 |
| **Plan 3:** | | | | |
| 1 | 8,000 | 800 | 1,724 | 2,524 |
| 2 | 6,276 | 628 | 1,896 | 2,524 |
| 3 | 4,380 | 438 | 2,086 | 2,524 |
| 4 | 2,294 | 230 | 2,294 | 2,524 |
| Total: | 20,960 $-yr | 2,096 | 8,000 | 10,096 |
| **Plan 4:** | | | | |
| 1 | 8,000 | 800 | 0 | 0 |
| 2 | 8,800 | 880 | 0 | 0 |
| 3 | 9,680 | 968 | 0 | 0 |
| 4 | 10,648 | 1,065 | 8,000 | 11,713 |
| Total: | 37,130 $-yr | 3,713 | 8,000 | 11,713 |

Table 1: **4 repayment plans.** $8,000 principal, 10% annual interest, 4 years. (Transp.)

[6] Compound interest: rebeet de'rebeet, rebeet tzvurah.

# 4 Interest Formulas for Present and Future Equivalent Values

## 4.1 Single Loan or Investment

§ **Primary source:** DeGarmo *et al,* pp.73–77.

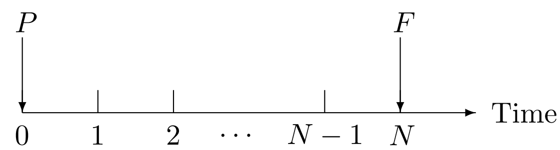


Figure 1: Cash flow program, section 4.1.

§ **Cash flow program,** fig. 1:

- Single **present** sum $P$: loan or investment at time $t = 0$.
- Single **future** sum $F$.
- $N$ periods.
- $i$: Interest rate (for loan) or profit rate (for investment).

§ **Find $F$ given $P$:**

- After 1 period: $F = (1+i)P$.
- After 2 periods: $F = (1+i)[(1+i)P] = (1+i)^2 P$.
- After $N$ periods:

$$F = (1+i)^N P \tag{4}$$

§ **Find $P$ given $F$.** Invert eq.(4):

$$P = \frac{1}{(1+i)^N} F \tag{5}$$

## 4.2 Constant Loan or Investment

§ **Primary source:** DeGarmo *et al,* pp.78–85.

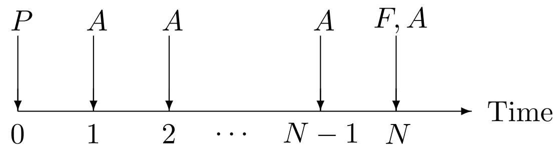


Figure 2: Cash flow program, section 4.2.

§ **Cash flow program,** fig. 2:

- $A$: An **annual** loan, investment or profit, occurring at the **end of each period.** (Sometimes called annuity)[7]
- $i$: Interest rate (for loan) or profit rate (for investment).
- $N$ periods.

§ **Equivalent present, annual and future sums:**

- Given $A$, $N$ and $i$, find:
  - Future equivalent sum $F$ occurring at the same time as the last $A$, at **end of period** $N$.
    (Section 4.2.1, p.9.)
  - Present equivalent sum $P$:
    loan or investment occurring 1 period before first constant amount $A$.
    (Section 4.2.2, p.10.)
- Given $P$, $N$ and $i$, find:
  - Annual equivalent sum $A$ occuring at end of each period.
    (Section 4.2.3, p.11.)

[7]Annuity: Kitzvah shnatit.

### 4.2.1 Find $F$ given $A$, $N$ and $i$

§ **Motivation:**

- Make $N$ annual deposits of $A$ dollars at end of each year.
- Annual interest is $i$.
- How much can be withdrawn at end of year $N$?

§ **Motivation:**

- Earn $N$ annual profits of $A$ dollars at end of each year.
- Re-invest at profit rate $i$.
- How much can be withdrawn at end of year $N$?

§ **Sums of a geometric series** that we will use frequently, for $x \neq 1$:

$$\sum_{n=0}^{N-1} x^n = \frac{x^N - 1}{x - 1} \tag{6}$$

$$\sum_{n=1}^{N-1} x^n = \frac{x^N - x}{x - 1} \tag{7}$$

- Special case: $x = \frac{1}{1+i}$:

$$\sum_{n=0}^{N-1} \frac{1}{(1+i)^n} = \frac{1 - \frac{1}{(1+i)^N}}{1 - \frac{1}{1+i}} = \frac{1 + i - (1+i)^{-(N-1)}}{i} \tag{8}$$

$$\sum_{n=1}^{N-1} \frac{1}{(1+i)^n} = \frac{\frac{1}{1+i} - \frac{1}{(1+i)^N}}{1 - \frac{1}{1+i}} = \frac{1 - (1+i)^{-(N-1)}}{i} \tag{9}$$

§ **Find $F$ given $A$, $N$ and $i$:** Value of annuity plus interest after $N$ periods.

- From $N$th period: $(1+i)^0 A$. (Because last $A$ at end of last period.)
- From $(N-1)$th period: $(1+i)^0(1+i)A = (1+i)^1 A$.
- From $(N-2)$th period: $(1+i)^0(1+i)(1+i)A = (1+i)^2 A$.
- From $(N-n)$th period: $(1+i)^n A, \quad n = 0, \ldots, N-1$.
- After all $N$ periods:

$$F = (1+i)^0 A + (1+i)^1 A + (1+i)^2 A + \cdots + (1+i)^{N-1} A \tag{10}$$

$$= \sum_{n=0}^{N-1} (1+i)^n A \tag{11}$$

$$= \frac{(1+i)^N - 1}{i} A \tag{12}$$

§ **Example** of eq.(12), table 2, p.10 (transparency):

- Column 3: ratio of final worth, $F$, to annuity, $A$. Why does $F/A$ **increase** as $i$ **increases**?
- Column 4: effect of compound interest: $F > NA$. Note **highly non-linear** effect at long time.

| $N$ | $i$ | $F/A$ | $F/NA$ |
|---|---|---|---|
| 5 | 0.03 | 5.3091 | 1.0618 |
| 5 | 0.1 | 6.1051 | 1.2210 |
| 10 | 0.03 | 11.4639 | 1.1464 |
| 10 | 0.1 | 15.9374 | 1.5937 |
| 30 | 0.03 | 47.5754 | 1.5858 |
| 30 | 0.1 | 164.4940 | 5.4831 |

Table 2: Example of eq.(12). (Transp.)

#### 4.2.2 Find $P$ given $A$, $N$ and $i$

§ **Motivation:**

- Repair of a machine now would increase output by $20,000 at end of each year for 5 years.
- We can take a loan now at 7% interest to finance the repair.
- How large a loan can we take if we must cover it from accumulated increased earning after 5 years?

§ **Repayment of loan,** $P$, after $N$ years at interest $i$, from eq.(4), p.7:

$$F = (1+i)^N P \tag{13}$$

§ **The loan, $P$, must be equivalent to the annuity, $A$.** Hence:
**Eq.(13) must equal** accumulated value of increased yearly earnings, $A$, eq.(12), p.9:

$$F = \frac{(1+i)^N - 1}{i} A \tag{14}$$

§ Equate eqs.(13) and (14) and solve for $P$:

$$P = \frac{(1+i)^N - 1}{i(1+i)^N} A = \frac{1-(1+i)^{-N}}{i} A \tag{15}$$

- This is the largest loan we can cover from accumulated earnings.
- This is the present (starting time, $t = 0$) equivalent value of the annuity.

§ **Example** of eq.(15), table 3 (transparency):

• Column 3: ratio of loan, $P$, to annuity, $A$. Why does $P/A$ **decrease** as $i$ **increases,** unlike table 2?

• Column 4: effect of compound interest: $P < NA$.

| $N$ | $i$ | $P/A$ | $P/NA$ |
|---|---|---|---|
| 5 | 0.03 | 4.580 | 0.916 |
| 5 | 0.1 | 3.791 | 0.758 |
| 10 | 0.03 | 8.530 | 0.853 |
| 10 | 0.1 | 6.145 | 0.615 |
| 30 | 0.03 | 19.600 | 0.655 |
| 30 | 0.1 | 9.427 | 0.314 |

Table 3: Example of eq.(15). (Transp.)

### 4.2.3 Find $A$ given $P$, $N$ and $i$

§ $F$ and $A$ are related by eq.(12), p.9:

$$F = \frac{(1+i)^N - 1}{i} A \tag{16}$$

- Thus:

$$A = \frac{i}{(1+i)^N - 1} F \tag{17}$$

- $F$ and $P$ are related by eq.(4), p.7:

$$F = (1+i)^N P \tag{18}$$

- Thus $A$ and $P$ are related by:

$$A = \frac{i(1+i)^N}{(1+i)^N - 1} P \tag{19}$$

**Example 2** We can now explain Plan 3 in table 1, p.6.

- The principal is $P = 8,000$.
- The interest rate is $i = 0.1$.
- The number of periods is $N = 4$.
- Thus the equivalent equal annual payments, $A$, are from eq.(19):

$$A = \frac{0.1 \times 1.1^4}{1.1^4 - 1} 8,000 = 0.3154708 \times 8,000 = 2,523.77 \tag{20}$$

■

### 4.3 Variable Loan or Investment

§ **Cash flow program:**

- $A_1, A_2, \ldots, A_N$: Sequence of annual loans or investments, occurring at the **end of each period.**
- $i$: Interest rate (for loan) or profit rate (for investment).
- $N$ periods.

§ **Future equivalent sum:** Given $A_1, A_2, \ldots, A_N$ and $i$, find:

- Future equivalent sum $F$ occurring at the same time as $A_N$.
- Generalization of eq.(10) on p.9:
- From $N$th period: $(1+i)^0 A_N$.
- From $(N-1)$th period: $(1+i)^0(1+i)A_{N-1} = (1+i)^1 A_{N-1}$.
- From $(N-2)$th period: $(1+i)^0(1+i)(1+i)A_{N-2} = (1+i)^2 A_{N-2}$.
- From $(N-n)$th period: $(1+i)^n A_{N-n}, \quad n = 0, \ldots, N-1$.

$$F = (1+i)^0 A_{N-0} + (1+i)^1 A_{N-1} + (1+i)^2 A_{N-2} + \cdots + (1+i)^{N-1} A_{N-(N-1)} \tag{21}$$

$$= \sum_{n=0}^{N-1} (1+i)^n A_{N-n} \tag{22}$$

§ **Present equivalent sum:** Given $A_1, A_2, \ldots, A_N$ and $i$, find:

- Present equivalent sum $P$: loan or investment occurring 1 period before first amount $A_1$.
- Analogous to eqs.(13)–(15), p.10:
  - ○ **Repayment of loan,** $P$, after $N$ years at interest $i$, from eq.(4), p.7:

$$F = (1+i)^N P \tag{23}$$

  - ○ **This must equal** accumulated value of increased yearly earnings, eq.(22).
  - ○ Equate eqs.(22) and (23) and solve for $P$:

$$P = \frac{1}{(1+i)^N} \sum_{n=0}^{N-1} (1+i)^n A_{N-n} \tag{24}$$

$$= \sum_{n=0}^{N-1} (1+i)^{-(N-n)} A_{N-n} \tag{25}$$

— This is the largest loan we can cover from accumulated earnings.

— This is the present (starting time) equivalent value of the annuity.

## 4.4 Variable Interest, Loan or Investment

§ **Partial source:** DeGarmo *et al*, p.101.

§ **Cash flow program:**

- $A_1, A_2, \ldots, A_N$: Sequence of annual loans or investments,
  occurring at the end of each period.
- $i_1, i_2, \ldots, i_N$: Sequence of annual interest rates (for loan) or profit rates (for investment).
- $N$ periods.

§ **Future equivalent sum:** Given $A_1, A_2, \ldots, A_N$ and $i_1, i_2, \ldots, i_N$, find:

- Future equivalent sum $F$ occurring at the same time as $A_N$.
- Generalization of eqs.(21) and (22) on p.12:
- From $N$th period: $(1 + i_N)^0 A_N$.
- From $(N - 1)$th period: $(1 + i_N)^0(1 + i_{N-1})A_{N-1}$.
- From $(N - 2)$th period: $(1 + i_N)^0(1 + i_{N-1})(1 + i_{N-2})A_{N-2}$.
- From $(N - n)$th period: $(1 + i_N)^0(1 + i_{N-1}) \cdots (1 + i_{N-(n-1)})(1 + i_{N-n})A_{N-n}$,
  $n = 0, \ldots, N - 1$.

$$F = \sum_{n=0}^{N-1} \left( \prod_{k=1}^{n} (1 + i_{N-k}) \right) A_{N-n} \tag{26}$$

§ **Present equivalent sum:** Given $A_1, A_2, \ldots, A_N$ and $i_1, i_2, \ldots, i_N$, find:

- Present equivalent sum $P$: loan or investment occurring 1 period before first amount $A_1$.
- Analogous to eqs.(23)–(24), p.12:
  - **Repayment of loan,** $P$, after $N$ years at interest $i$, generalizing eq.(4), p.7:

$$F = \left( \prod_{k=0}^{N-1} (1 + i_{N-k}) \right) P \tag{27}$$

  - **This must equal** accumulated value of increased yearly earnings, eq.(26).
  - Equate eqs.(26) and (27) and solve for $P$:

$$P = \frac{\sum_{n=0}^{N-1} \left( \prod_{k=1}^{n} (1 + i_{N-k}) \right) A_{N-n}}{\prod_{k=0}^{N-1} (1 + i_{N-k})} \tag{28}$$

— This is the largest loan we can cover from accumulated earnings.
— This is the present (starting time) equivalent value of the annuity.

## 4.5 Compounding More Often Than Once per Year

**Example 3** (DeGarmo, p.105.)

- Statement:
  \$100 is invested for 10 years at *nominal* 6% interest per year, *compounded quarterly.*
  What is the Future Worth ($FW$) after 10 years?
- Solution 1:
  - 4 compounding periods per year. Total of $4 \times 10 = 40$ periods.
  - Interest rate per period is $(6\%)/4 = 1.5\%$ which means $i = 0.015$.
  - The $FW$ after 10 years is, from eq.(4), p.7:

$$F = (1+i)^N P = 1.015^{40} \times 100 = \$181.40 \tag{29}$$

- Solution 2:
  - What we mean by "compounded quarterly" is that
    the *effective annual interest rate* is defined by the following 2 relations:

$$i_{\text{qtr}} = i_{\text{nominal}}/4 \tag{30}$$

and

$$1 + i_{\text{ef ann}} = (1 + i_{\text{qtr}})^4 \quad \Longrightarrow \quad i_{\text{ef ann}} = (1 + 0.015)^4 - 1 = 0.061364 \tag{31}$$

  - Thus the effective annual interest rate is 6.1364%.
  - The $FW$ after 10 years is, from eq.(4), p.7:

$$F = 1.061364^{10} \times 100 = \$181.40 \tag{32}$$

- Why do eqs.(29) and (32) agree? The general solution will explain.

■

§ **General solution.**

- A sum $P$ is invested for $N$ years at
  nominal annual interest $i$ compounded $m$ equally spaced times per year.
- The interest rate per period is (generalization of eq.(30)):

$$i_{\text{per}} = \frac{i}{m} \tag{33}$$

- What we mean by "compounded $m$ times per year" is that
  the *effective annual interest rate* is determined by (generalization of eq.(31)):

$$1 + i_{\text{ef ann}} = (1 + i_{\text{per}})^m \tag{34}$$

- The $FW$ by the "period calculation" method is:

$$F_{\text{per}} = (1 + i_{\text{per}})^{mN} P \tag{35}$$

- The $FW$ by the "effective annual calculation" method is:

$$F_{\text{ef ann}} = (1 + i_{\text{ef ann}})^N P \tag{36}$$

- Combining eqs.(34)–(36) shows:

$$F_{\text{ef ann}} = F_{\text{per}} \tag{37}$$

**Example 4** § **Example.** (DeGarmo, p.105)

- $10,000 loan at nominal 12% annual interest for 5 years, compounded monthly.
- Equal end-of-month payments, $A$, for 5 years.
- What is the value of $A$?
- Solution:
  - ◦ The period interest, eq.(33), p.14, is $i = 0.12/12 = 0.01$.
  - ◦ The principle, $P = 10,000$, is related to equal monthly payments $A$ by eq.(19), p.11:

$$\begin{aligned} A &= \frac{i(1+i)^N}{(1+i)^N - 1} P && (38) \\ &= 0.0222444P && (39) \\ &= \$222.44 && (40) \end{aligned}$$

- Why is the following calculation **not correct?**
  - ◦ The $FW$ of the loan is:

$$FW = 1.01^{5\times 12} P = 1.816697 \times 10,000 = 18,166.97 \quad (41)$$

  - ◦ Divide this into 60 equal payments:

$$A' = \frac{18,166.97}{60} = \$302.78 \quad (42)$$

  - ◦ Eq.(41) is correct.
  - ◦ Eq.(42) is **wrong** because it takes a **final worth** and charges it at earlier times, ignoring the **equivalent value** of these intermediate payments.
    This explains why $A' > A$.

■

# Part II
# Applications of Time-Money Relationships

§ **The problem:**

- Given several different design concepts, each technologically acceptable.
- Select one option or prioritize all the options.

§ **The economic approach:**

- Treat each option as a *capital investment.*
- Consider:
  - *Expenditures* for implementation.
  - *Revenues* or *savings* over time.
  - Attractive or acceptable *return on investment.*

§ We will consider **two time-value methods:**

- Present Worth, section 5, p.17.
- Future Worth, section 6, p.20.
- We will show that these are **equivalent.**

§ Central idea: **Minimum Attractive Rate of Return** (MARR):[8]

- The MARR is an interest rate or profit rate.
- Subjective judgment.
- Least rate of return from other known alternatives.
- Examples: DeGarmo pp.141–143.

[8] Shiur ha'revach ha'kvil ha'minimali.

# 5 Present Worth Method

§ **Primary source:** DeGarmo *et al,* pp.144–149.

§ **Basic idea** of present worth ($PW$):

- Evaluate present worth (net present value) of all cash flows (cost and revenue), based on an interest rate usually equal to the MARR.
- The $PW$ is the profit left over after the investment.
- We assume that cash revenue is invested at interest rate equal to the MARR.
- The $PW$ is also called Net Present Value (NPV).

§ **Basic Formula** for calculating the $PW$.

- $i$ = interest rate, e.g. MARR.
- $F_k$ = cash flow in **end of periods** $k = 0, 1, , \ldots, N$. Positive for revenue, negative for cost. $F_0$ = initial investment at **start** of the $k = 1$ period.
- $N$ = number of periods.
- Basic relation, eq.(5), p.7, $PW$ of revenue $F_k$ at period $k$:

$$P_k = \frac{1}{(1+i)^k} F_k \tag{43}$$

- Formula for calculating the $PW$ of revenue stream $F_0, F_1, \ldots, F_N$:

$$\begin{aligned} PW &= (1+i)^{-0}F_0 + (1+i)^{-1}F_1 + \cdots + (1+i)^{-k}F_k + \cdots + (1+i)^{-N}F_N && (44) \\ &= \sum_{k=0}^{N} (1+i)^{-k} F_k && (45) \end{aligned}$$

- For a constant revenue stream, $F, F, \ldots, F$ from $k = 0$ to $k = N$:

$$\begin{aligned} PW &= \sum_{k=0}^{N} (1+i)^{-k} F && (46) \\ &= \frac{\left(\frac{1}{1+i}\right)^{N+1} - 1}{\frac{1}{1+i} - 1} F && (47) \\ &= \frac{1 + i - (1+i)^{-N}}{i} F && (48) \end{aligned}$$

**Example 5** Does the Present Worth method justify the following project?

- $S$ = Initial cost of the project = \$10,000.
- $R_k$ = revenue at the end of $k$th period = \$5,310.
- $C_k$ = operating cost at the end of $k$th period = \$3,000.
- $N$ = number of periods.
- $M$ = re-sale value of equipment at end of project = \$2,000.
- MARR = 10%, so $i = 0.1$.
- Adapting eq.(45), p.17, the $PW$ is:

$$\begin{aligned} PW &= -S + \sum_{k=1}^{N} (1+i)^{-k} R_k - \sum_{k=1}^{N} (1+i)^{-k} C_k + (1+i)^{-N} M && (49) \\ &= -10,000 + 3.7908 \times 5,310 - 3.7908 \times 3,000 + 0.6209 \times 2,000 && (50) \\ &= -10,000 + 20,129.15 - 11,372.40 + 1,241.80 && (51) \\ &= -\$1.41 && (52) \end{aligned}$$

- The project essentially breaks even (it loses \$1.41), so it is marginally justified by $PW$. ■

§ **Bonds:**[9] **General formulation.**[10]

- Bonds and stocks[11] are both securities:[12]
  *Bonds:* a loan to the firm. *Stocks:* equity or partial ownership of firm.
- $F$ = face value (putative purchase cost) of bond.
- $r$ = bond rate = interest paid by bond at end of each period.
- $C = rF$ = coupon payment (periodic interest payment) at end of each period.
- $M$ = market value of bond at maturity; usually equals $F$.
- $i$ = discount rate[13] at which the sum of all future cash flows from the bond (coupons and principal) are equal to the price of the bond. May be the MARR.
- Note: $r$ and $i$ are **different** though they are both rates (percents) of a sum:
  - ○ $r$ is the profit from the bond.
  - ○ $i$ assesses the time-value of this profit.
- $N$ = number of periods.
- Formula for calculating a bond's price.[14] This is the $PW$ of the bond:

$$P = (1+i)^{-N}M + \sum_{k=1}^{N}(1+i)^{-k}C \tag{53}$$

$$= (1+i)^{-N}M + \frac{1-(1+i)^{-N}}{i}C \tag{54}$$

**Example 6 Bonds.**[15]

- $F$ = face value = \$5,000.
- $r$ = bond rate = 8% paid annually at end of each year.
- Bond will be redeemed at face value after 20 years, so $M = F$ and $N = 20$.
- (a) How much should be paid now to receive a yield of 10% per year on the investment?
  $C = 0.08 \times 5,000 = 400$. $M = 5,000$. $i = 0.1$, so from eq.(54):

$$P = 1.1^{-20}5000 + \frac{1-1.1^{-20}}{0.1}400 \tag{55}$$

$$= 0.1486 \times 5,000 + 8.5135 \times 400 \tag{56}$$

$$= 743.00 + 3,405.43 \tag{57}$$

$$= 4,148.43 \tag{58}$$

- (b) If this bond is purchased now for \$4,600, what annual yield would the buyer receive?
  We must numerically solve eq.(54) for $i$ with $P, M, N$ and $C$ given:

$$4,600 = (1+i)^{-20}5000 + \frac{1-(1+i)^{-20}}{i}400 \tag{59}$$

The result is about 8.9% per year, which is less than 10% because $4,600 > 4,148.43$. ■

---

[9] Igrot hov. "Igrot" is written with alef.
[10] http://en.wikipedia.org/wiki/Bond_(finance)
[11] miniot.
[12] niyarot erech.
[13] Discount rate: hivun. "hivun" is written with 2 vav's.
[14] http://en.wikipedia.org/wiki/Bond_valuation
[15] DeGarmo, p.148.

**Example 7** (DeGarmo, pp.168–170).

- Project definition:
  - $P$ = initial investment = \$140,000.
  - $R_k$ = revenue at end of $k$th year = $\frac{2}{3}(45,000 + 5,000k)$.
  - $C_k$ = operating cost paid at end of $k$th year = \$10,000.
  - $M_k$ = maintenance cost paid at end of $k$th year = \$1,800.
  - $T_k$ = tax and insurance paid at end of $k$th year = $0.02P = 2,800$.
  - $i$ = MARR interest rate = 15%.
- Goal: recover investment with interest at the MARR after $N = 10$ years.
- Question: Should the project be launched?
- Solution:
  - Evaluate the $PW$.
  - Launch project if $PW$ is positive.
  - (What about risk and uncertainty?)
  - Adapting the $PW$ relation, eq.(45), p.17:

$$
\begin{aligned}
PW &= -P + \sum_{k=1}^{N} (R_k - C_k - M_k - T_k)(1+i)^{-k} && (60) \\
&= -140,000 + \sum_{k=1}^{10} \left( \frac{2}{3}(45,000 + 5,000k) - 10,000 - 1,800 - 2,800 \right) 1.15^{-k} && (61) \\
&= \$10,619 && (62)
\end{aligned}
$$

  - The $PW$ is positive so, ignoring risk and uncertainty, the project is justified. ■

# 6 Future Worth Method

§ **Primary source:** DeGarmo *et al,* pp.149–150.

§ **Basic idea** of future worth ($FW$):

- Evaluate equivalent worth of all cash flows (cost and revenue) at end of planning horizon, based on an interest rate usually equal to the MARR.
- The $FW$ is equivalent to the $PW$.

§ **Basic Formula** for calculating the $FW$.

- $i$ = interest rate, e.g. MARR.
- $F_k$ = cash flow in **end of periods** $k = 0, 1, , \ldots, N$. Positive for revenue, negative for cost. $F_0$ = initial investment at **start** of the $k = 1$ period.
- $N$ = number of periods.
- Basic relation, eq.(4), p.7, $FW$ at end of planning horizon, of revenue $F_k$ at end of period $k$:

$$FW_k = (1+i)^{N-k} F_k \tag{63}$$

- Formula for calculating the $FW$ of revenue stream $F_0, F_1, \ldots, F_N$:

$$FW = (1+i)^N F_0 + (1+i)^{N-1} F_1 + \cdots + (1+i)^{N-k} F_k + \cdots + (1+i)^0 F_N \tag{64}$$

$$= \sum_{k=0}^{N} (1+i)^{N-k} F_k \tag{65}$$

- The relation between $PW$ and $FW$, eq.(5), p.7:

$$PW = (1+i)^{-N} FW \tag{66}$$

$$= (1+i)^{-N} \sum_{k=0}^{N} (1+i)^{N-k} F_k \tag{67}$$

$$= \sum_{k=0}^{N} (1+i)^{-k} F_k \tag{68}$$

which is eq.(45), p.17.

§ Eq.(66) shows that PW and FW are equivalent for ranking alternative projects, though numerically they are different.

**Example 8**

- $F_0 = \$25,000 =$ cost of new equipment.
- $F_k = \$8,000$ net revenue (after operating cost), $k = 1, \ldots, 5$.
- $i = 0.2 = 20\%$ MARR.
- $N = 5 =$ planning horizon.
- $M = \$5,000 =$ market value of equipment at end of planning horizon.
- Adapting eq.(65), p.20, the $FW$ is:

$$
\begin{aligned}
FW &= \sum_{k=0}^{N} (1+i)^{N-k} F_k + M &(69)\\
&= \underbrace{-(1.2)^5 \times 25,000}_{\text{step } k=0} + \underbrace{\sum_{k=0}^{4} 1.2^k \times 8,000}_{\text{steps } k=5,\ \ldots,\ 1} + 5,000 &(70)\\
&= -1.2^5 \times 25,000 + \frac{1.2^5 - 1}{1.2 - 1} \times 8,000 + 5,000 &(71)\\
&= -62,208 + 59,532.80 + 5,000 &(72)\\
&= 2,324.80 &(73)
\end{aligned}
$$

- This project is profitable.
- The $PW$ of this project is:

$$
\begin{aligned}
PW &= (1+i)^{-N} FW &(74)\\
&= (1.2)^{-5} \times 2,324.80 &(75)\\
&= 934.28 &(76)
\end{aligned}
$$

■

# Part III
# Implications of Uncertainty

§ **Sources of uncertainty:**

- The **future** is uncertain:
  - Costs.
  - Revenues.
  - Interest rates.
  - Technological innovations.
  - Social and economic changes or upheavals.
- The **present** is uncertain:
  - Capabilities.
  - Goals.
  - Opportunities.
- The **past** is uncertain:
  - Biased or incomplete historical data.
  - Limited understanding of past processes, successes and failures.

## 7 Uncertain Profit Rate, $i$, of a Single Investment, $P$

§ **Background:** section 4.1, p.7.

## 7.1 Uncertainty

§ **Problem statement:**

- $P$ = investment now.
- $i$ = projected profit rate, %/year.
- $FW$ = future worth:

$$FW = (1+i)^N P \tag{77}$$

- *Questions:*
  - Is the investment worth it?
  - Is the $FW$ good enough? Is $FW$ at least as large as $FW_c$?

$$FW(i) \geq FW_c \tag{78}$$

- *Problem:* $i$ highly uncertain.
- *Question:* How to choose the value of $FW_c$?

§ **The info-gap.**

- $\widetilde{i}$ = **known** estimate of profit rate.
- $i$ = **unknown** but constant true profit rate. Why is assumption of constancy important? Eq.(77)
- $s$ = known estimate of error of $\widetilde{i}$. $i$ may err by $s$ or more. **Worst case not known.**
- Fractional error:

$$\left| \frac{i-\widetilde{i}}{s} \right| \tag{79}$$

- Fractional error is **bounded:**

$$\left| \frac{i-\widetilde{i}}{s} \right| \leq h \tag{80}$$

- The bound, $h$, is **unknown:**

$$\left| \frac{i-\widetilde{i}}{s} \right| \leq h, \quad h \geq 0 \tag{81}$$

- **Fractional-error info-gap model** for uncertain profit rate:[16]

$$\mathcal{U}(h) = \left\{ i : \ \left| \frac{i-\widetilde{i}}{s} \right| \leq h \right\}, \quad h \geq 0 \tag{82}$$

  - Unbounded family of nested sets of $i$ values.
  - No known worst case.
  - No known probability distribution.
  - $h$ is the **horizon of uncertainty.**

§ **The question:** Is the $FW$ good enough? Is $FW$ at least as large as a critical value $FW_c$?

$$FW(i) \geq FW_c \tag{83}$$

- Can we answer this question? No, because $i$ is unknown.
- What (useful) question can we answer?

[16] There are other constraints on an interest rate, $i$, but we won't worry about them now.

## 7.2 Robustness

§ **Robustness question** (that we *can* answer): How large an error in $\widetilde{i}$ can we tolerate?

§ **Robustness function:**

$$\widehat{h}(FW_{\rm c}) = \text{maximum tolerable uncertainty} \tag{84}$$
$$= \text{maximum } h \text{ such that } FW(i) \geq FW_{\rm c} \text{ for all } i \in \mathcal{U}(h) \tag{85}$$
$$= \max \left\{ h : \left( \min_{i \in \mathcal{U}(h)} FW(i) \right) \geq FW_{\rm c} \right\} \tag{86}$$

§ **Evaluating the robustness:**

- Inner minimum:

$$m(h) = \min_{i \in \mathcal{U}(h)} FW(i) \tag{87}$$

- $m(h)$ vs $h$:
  - Decreasing function. **Why?**
  - From eq.(77) ($FW = (1+i)^N P$) and IGM in eq.(82), p.23: $m(h)$ occurs at $i = \widetilde{i} - sh$:[17]

$$m(h) = (1 + \widetilde{i} - sh)^N P \tag{88}$$

- What is greatest tolerable horizon of uncertainty, $h$? Equate $m(h)$ to $FW_{\rm c}$ and solve for $h$:

$$(1 + \widetilde{i} - sh)^N P = FW_{\rm c} \quad \Longrightarrow \quad \boxed{\widehat{h}(FW_{\rm c}) = \frac{1+\widetilde{i}}{s} - \frac{1}{s}\left(\frac{FW_{\rm c}}{P}\right)^{1/N}} \tag{89}$$

§ **Properties of the robustness curve:** (See fig. 3)

- **Trade off:** robustness **up** (good) only for $FW_{\rm c}$ **down** (bad). (Pessimist's theorem)
- **Zeroing:** no robustness of predicted $FW$: $(1+\widetilde{i})^N P$.

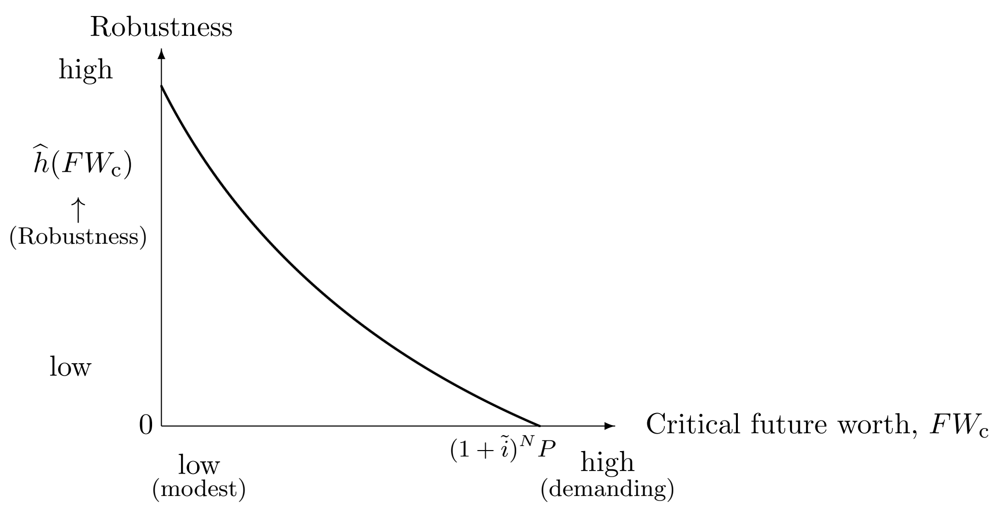


Figure 3: Robustness curve.

[17] This allows $1 - i < 0$ which may not be allowed or meaningful. However, we will see that $1 - i \geq 0$ for all $h \leq \widehat{h}$.

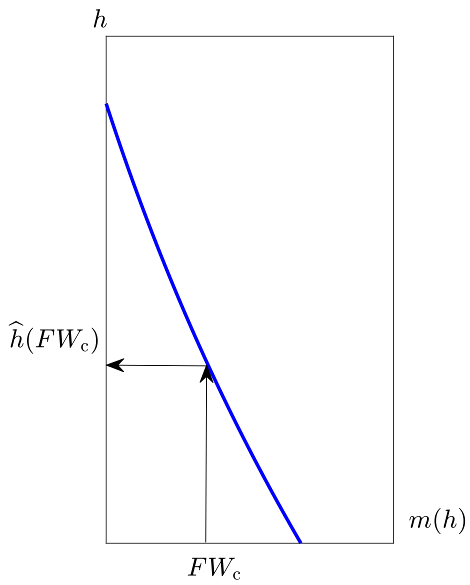


Figure 4: $m(h)$ is inverse function of $\widehat{h}(FW_c)$.

§ We understand from fig. 4 that $m(h)$ is the **inverse function** of $\widehat{h}(FW_c)$. **Why?**

§ This is important because sometimes we can only calculate $m(h)$ but not its inverse: $\widehat{h}(FW_c)$.

### 7.3 Decision Making and the Innovation Dilemma

**§ Decision making.**

- Suppose your information is something like:
  - Annual profits are typically about 12%, plus or minus 2% or 4% or more, or,
  - Similar projects have had average profits of 12% with standard deviation of 3%, but the future is often surprising.
- You might quantify this information with an info-gap model like eq.(82), p.23 with $\widetilde{i} = 0.12$ and $s = 0.03$.
- You might then construct the robustness function like eq.(89), p.24.
- What $FW_{\rm c}$ is credible? One with no less than "several" units of robustness.
- For instance, from eq.(89):

$$\widehat{h}(FW_{\rm c}) \approx 3 \quad \Longrightarrow \quad \frac{FW_{\rm c}}{P} \approx (1 + \widetilde{i} - 3s)^N \qquad (90)$$

With $\widetilde{i} = 0.12$, $s = 0.03$, $N = 10$ years this is:

$$\widehat{h}(FW_{\rm c}) = 3 \quad \Longrightarrow \quad \frac{FW_{\rm c}}{P} = (1 + 0.12 - 3 \times 0.03)^{10} = 1.03^{10} = 1.34 \qquad (91)$$

- Compare with the nominal profit ratio predicted with the best estimate, eq.(77), p.23:

$$\frac{FW_{\rm c}(\widetilde{i})}{P} = (1 + \widetilde{i})^N = (1.12)^{10} = 3.11 \qquad (92)$$

- Given the knowledge and the info-gap, a credible profit ratio is

    1.34 (robustness = 3)

  rather than

    3.11 (robustness = 0).

**§ Innovation dilemma.**

- Choose between two projects or design concepts:
  - State of the art, with standard projected profit and moderate uncertainty.
  - New and innovative, with higher projected profit and higher uncertainty.
- For instance:
  - SotA: $\widetilde{i} = 0.03$, $s = 0.015$, $N = 10$. So $FW(\widetilde{i})/P = (1 + \widetilde{i})^{10} = 1.34$.
  - Innov: $\widetilde{i} = 0.05$, $s = 0.04$, $N = 10$. So $FW(\widetilde{i})/P = (1 + \widetilde{i})^{10} = 1.63$.
- The dilemma:

  Innovation is predicted to be better, but it is more uncertain and thus may be worse.
- Robustness functions shown in fig. 5, p.27.
- Note trade off and zeroing.
- SotA more robust for $FW_{\rm c}/P < 1.2$. Note: $\widehat{h}(FW_{\rm c}/P = 1|\text{SotA}) = 2$.
- Innov more robust for $FW_{\rm c}/P > 1.2$. Note: $\widehat{h}(FW_{\rm c}/P > 1.2|\text{innov}) < 1$.
- Neither option looks reliably attractive.
- Generic analysis:
  - Cost of robustness: slope: Greater cost of robustness for innovative option.
  - Innovative option putatively better, but greater cost of robustness.
  - Result: preference reversal.

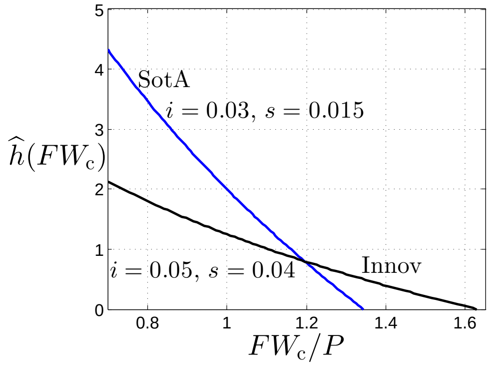


Figure 5: Illustration of the innovation dilemma. (Transp.)

# 8 Uncertain Constant Yearly Profit, $A$

§ **Background:** section 4.2, p.8.

## 8.1 Info-Gap on $A$

§ **Future worth** of constant profit:

- $A$ = profit at end of each period. E.g. annuity; no initial investment.
- $i$ = reinvest at profit rate $i$.
- $N$ = number of periods.
- The future worth is (eq.(12), p.9):

$$FW = \frac{(1+i)^N - 1}{i} A \tag{93}$$

§ **Uncertainty:** the constant end-of-period profit, $A$, is uncertain.

- $\widetilde{A}$ = known estimated profit.
- $A$ = unknown but constant true profit.
- $s_A$ = error of estimate. $A$ may be more or less that $\widetilde{A}$. **No known worst case.**
- Fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ A : \ \left| \frac{A - \widetilde{A}}{s_A} \right| \le h \right\}, \quad h \ge 0 \tag{94}$$

§ **Robust satisficing:**

- Satisfy performance requirement:

$$FW(A) \ge FW_{\rm c} \tag{95}$$

- Maximize robustness to uncertainty.

§ **Robustness:**

$$\widehat{h}(FW_{\rm c}) = \max \left\{ h : \ \left( \min_{A \in \mathcal{U}(h)} FW(A) \right) \ge FW_{\rm c} \right\} \tag{96}$$

§ **Evaluating the robustness:**

- Inner minimum:

$$m(h) = \min_{A \in \mathcal{U}(h)} FW(A) \tag{97}$$

- $m(h)$ vs $h$:
  - Decreasing function. **Why?**
  - Inverse of $\widehat{h}(FW_{\rm c})$. **Why?**
  - From eq.(93) ($FW = \sum_{k=0}^{N}(1+i)^{N-k}A = \frac{(1+i)^N-1}{i}A$), minimum occurs at $A = \widetilde{A} - sh$:

$$m(h) = \frac{(1+i)^N - 1}{i}(\widetilde{A} - s_A h) \tag{98}$$

- Equate to $FW_{\rm c}$ and solve for $h$:

$$\frac{(1+i)^N - 1}{i}(\widetilde{A} - s_A h) = FW_{\rm c} \quad \Longrightarrow \quad \boxed{\widehat{h}(FW_{\rm c}) = \frac{\widetilde{A}}{s_A} - \frac{i}{[(1+i)^N - 1]s_A}FW_{\rm c}} \tag{99}$$

  Or zero if this is negative.
- Zeroing and trade off. See fig. 6.

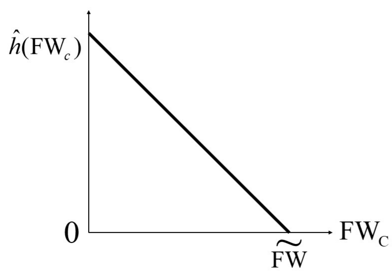


Figure 6: Trade off and zeroing of robustness.

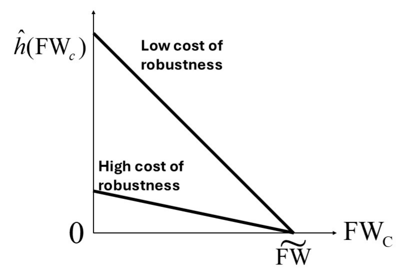


Figure 7: Low and High cost of robustness.

§ Consider the **cost of robustness,** determined by the slope of the robustness curve.

- Explain the **meaning** of cost of robustness. See fig. 7.

$$\text{slope} = -\frac{i}{[(1+i)^N - 1]s_A} = -\frac{1}{s_A}\left(\sum_{n=0}^{N-1}(1+i)^n\right)^{-1} \tag{100}$$

  Latter equality based on eq.(12), p.9.
- We see that:

$$\frac{\partial|\text{slope}|}{\partial s_A} < 0 \tag{101}$$

  This means that cost of robustness **increases** as uncertainty, $s_A$, **increases.** Why?
- We see that:

$$\frac{\partial|\text{slope}|}{\partial i} < 0 \tag{102}$$

  This means that cost of robustness **increases** as profit rate, $i$, **increases.** Why?
  From eq.(93) ($FW = \sum_{k}^{N}(1+i)^{N-k}A$): large $i$ magnifies $A$, thus magnifying uncertainty in $A$.
- Example. $i = 0.15$, $s_A = 0.05$, $N = 10$. Thus:

$$\text{slope} = \frac{0.15}{(1.15^{10} - 1)0.05} = 0.98 \ (\approx 1) \tag{103}$$

  Thus **decreasing** $FW_{\rm c}$ by 1 unit, **increases** the robustness by 1 unit.

## 8.2 PDF of $A$

§ **PDF: Probability Density Function.**

§ **Future worth** of constant profit, eq.(12), p.9:

- $A$ = profit (e.g. annuity) at end of each period.
- $i$ = reinvest at profit rate $i$.
- $N$ = number of periods.
- The future worth is:

$$FW(A) = \frac{(1+i)^N - 1}{i} A \tag{104}$$

§ **Requirement:**

$$FW(A) \geq FW_{\mathrm{c}} \tag{105}$$

§ **Problem:**

- $A$ is a random variable (but constant in time) with probability density function (pdf) $p(A)$.
- Is the investment reliable?

§ **Solution:** Use probabilistic requirement.

- Probability of failure:

$$P_{\mathrm{f}} = \mathrm{Prob}(FW(A) < FW_{\mathrm{c}}) \tag{106}$$

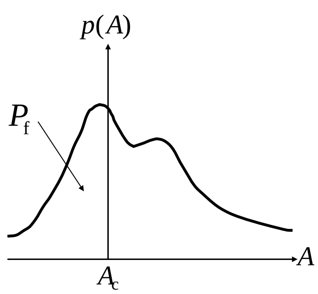


Figure 8: Probability of failure, eq.(112).

- Probabilistic requirement:

$$P_{\mathrm{f}} \leq P_{\mathrm{c}} \tag{107}$$

§ **Probability of failure for normal distribution:** $A \sim \mathcal{N}(\mu, \sigma^2)$

- The pdf:

$$p(A) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(A-\mu)^2}{2\sigma^2}\right) \tag{108}$$

- Probability of failure:

$$\begin{aligned}
P_{\mathrm{f}} &= \mathrm{Prob}(FW(A) < FW_{\mathrm{c}}) && (109)\\
&= \mathrm{Prob}\left(\frac{(1+i)^N - 1}{i} A \leq FW_{\mathrm{c}}\right) && (110)\\
&= \mathrm{Prob}\left(A \leq \underbrace{\frac{i}{(1+i)^N - 1} FW_{\mathrm{c}}}_{A_{\mathrm{c}}}\right) && (111)\\
&= \mathrm{Prob}\left(A \leq A_{\mathrm{c}}\right) && (112)\\
&= \mathrm{Prob}\left(\frac{A-\mu}{\sigma} \leq \frac{A_{\mathrm{c}}-\mu}{\sigma}\right) && (113)
\end{aligned}$$

- $\frac{A-\mu}{\sigma}$ is a standard normal variable, $\mathcal{N}(0,1)$, with cdf $\Phi(\cdot)$.
- Thus:

$$P_{\mathrm{f}} = \Phi\left(\frac{A_{\mathrm{c}}-\mu}{\sigma}\right) \tag{114}$$

$$= \Phi\left(\frac{i}{\sigma[(1+i)^N-1]}FW_{\mathrm{c}} - \frac{\mu}{\sigma}\right) \tag{115}$$

**Example 9**

- $FW_{\mathrm{c}} = \varepsilon FW(\mu)$. E.g. $\varepsilon = 0.5$.
- From eqs.(104) and (115):

$$P_{\mathrm{f}} = \Phi\left(\frac{\varepsilon\mu}{\sigma} - \frac{\mu}{\sigma}\right) = \Phi\left(-\frac{(1-\varepsilon)\mu}{\sigma}\right) \tag{116}$$

- From figs. 9 and 10 on p.30:
  - $P_{\mathrm{f}}$ increases as critical future worth increases (e.g. as $\varepsilon$ increases): $FW_{\mathrm{c}} = \varepsilon FW(\mu)$.
  - $P_{\mathrm{f}}$ increases as relative uncertainty increases: as $\mu/\sigma$ decreases.

■

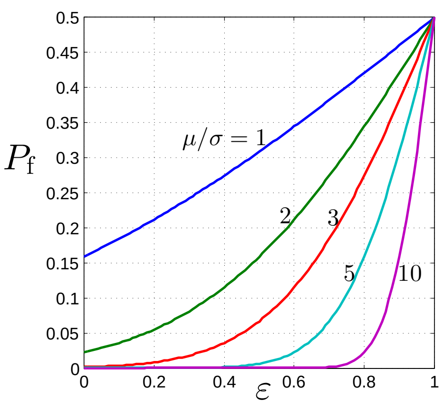


Figure 9: Probability of failure, eq.116. (Transp.)

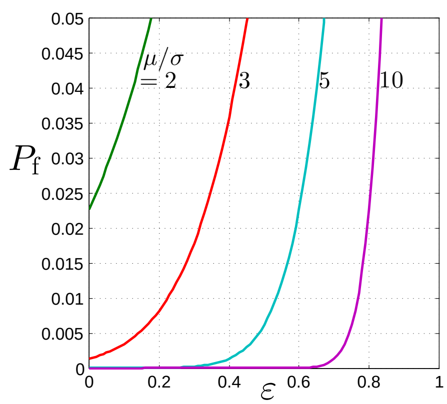


Figure 10: Probability of failure, eq.116. (Transp.)

## 8.3 Info-Gap on PDF of $A$

§ **Future worth** of constant profit, eq.(12), p.9:

- $A$ = profit (e.g. annuity) at end of each period.
- $i$ = reinvest at profit rate $i$.
- $N$ = number of periods.
- The future worth is:

$$FW(A) = \frac{(1+i)^N - 1}{i} A \tag{117}$$

§ **Requirement:**

$$FW(A) \geq FW_{\mathrm{c}} \tag{118}$$

§ **First Problem:**

- $A$ is a random variable (but constant in time) with probability density function (pdf) $p(A)$.
- Is the investment reliable?

§ **Solution:** Use probabilistic requirement.

- Probability of failure:

$$P_{\mathrm{f}} = \mathrm{Prob}(FW(A) < FW_{\mathrm{c}}) \tag{119}$$

$$= \mathrm{Prob}(A \leq A_{\mathrm{c}}) \tag{120}$$

$A_{\mathrm{c}} = \dfrac{i}{(1+i)^N - 1} FW_{\mathrm{c}}$, defined in eq.(111), p.29.

- Probabilistic requirement:

$$P_{\mathrm{f}} \leq P_{\mathrm{c}} \tag{121}$$

§ **Second problem:** pdf of $A$, $p(A)$, is info-gap uncertain with info-gap model $\mathcal{U}(h)$.

§ **Solution:** Embed the probabilistic requirement in an info-gap analysis of robustness to uncertainty.

§ **Robustness:**

$$\widehat{h}(P_{\mathrm{c}}) = \max \left\{ h : \ \left( \max_{p \in \mathcal{U}(h)} P_{\mathrm{f}}(p) \right) \leq P_{\mathrm{c}} \right\} \tag{122}$$

**Example 10 Normal distribution with uncertain mean.**

§ **Formulation:**

- $A \sim \mathcal{N}(\mu, \sigma^2)$.
- $\widetilde{\mu}$ = known estimated mean.
- $\mu$ = unknown true mean.
- $s_\mu$ = error estimate. $\mu$ may err more or less than $s_\mu$.
- Info-gap model:

$$\mathcal{U}(h) = \left\{ \mu : \ \left| \frac{\mu - \widetilde{\mu}}{s_\mu} \right| \leq h \right\}, \quad h \geq 0 \tag{123}$$

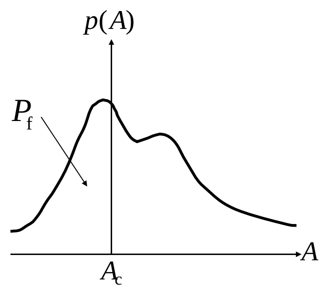


Figure 11: Probability of failure, eq.(120).

§ **Evaluating the robustness:**

- $M(h)$ = inner maximum in eq.(122).
- $M(h)$ occurs if $p(A)$ is shifted maximally left (fig. 11, p.32), so $\mu = \widetilde{\mu} - s_\mu h$:

$$M(h) = \max_{p \in \mathcal{U}(h)} \mathrm{Prob}(A \le A_c | \mu) \tag{124}$$

$$= \mathrm{Prob}\left( \frac{A - (\widetilde{\mu} - s_\mu h)}{\sigma} \le \frac{A_c - (\widetilde{\mu} - s_\mu h)}{\sigma} \,\middle|\, \mu = \widetilde{\mu} - s_\mu h \right) \tag{125}$$

$$= \Phi\left( \frac{A_c - (\widetilde{\mu} - s_\mu h)}{\sigma} \right) \tag{126}$$

$$= \Phi\left( \frac{i}{\sigma[(1+i)^N - 1]} FW_c - \frac{\widetilde{\mu} - s_\mu h}{\sigma} \right) \tag{127}$$

because $\frac{A-(\widetilde{\mu}-s_\mu h)}{\sigma}$ is standard normal.

- Let $FW_c = \varepsilon FW(\widetilde{\mu}) = \varepsilon \frac{(1+i)^N - 1}{i} \widetilde{\mu}$. Eq.(127) is:

$$M(h) = \Phi\left( \frac{\varepsilon \widetilde{\mu}}{\sigma} - \frac{\widetilde{\mu} - s_\mu h}{\sigma} \right) \tag{128}$$

$$= \Phi\left( -\frac{(1-\varepsilon)\widetilde{\mu} - s_\mu h}{\sigma} \right) \tag{129}$$

- $M(h)$ is the inverse of $\widehat{h}(P_c)$:
  $M(h)$ horizontally vs $h$ vertically is equivalent to $P_c$ horizontally vs $\widehat{h}(P_c)$ vertically.
  See figs. 12 and 13.
- **Zeroing:** $\widehat{h}(P_c) = 0$ when $P_c = P_f(\widetilde{\mu})$.
  Estimated probability of failure, $P_f(\widetilde{\mu})$, **increases** as relative error, $\sigma/\mu$, **increases**.
- **Trade off:** robustness decreases (gets worse) as $P_c$ decreases (gets better).
- **Cost of robustness:** increase in $P_c$ required to obtain given increase in $\widehat{h}$.
  Cost of robustness **increases** as $\sigma/\mu$ and $\sigma/s_\mu$ increase **at low** $P_c$; fig. 13.
- $P_f(\widetilde{\mu})$ and cost of robustness **change in reverse directions** as $\sigma/\mu$ changes.
  - This causes curve-crossing and preference-reversal.
  - At small $P_c$ (fig. 13): robustness increases as relative error, $\sigma/\mu$, falls (as $\frac{\mu}{\sigma}$ rises.)
  - At large $P_c$ (fig. 12): preference reversal at $P_c = 0.5$.

■

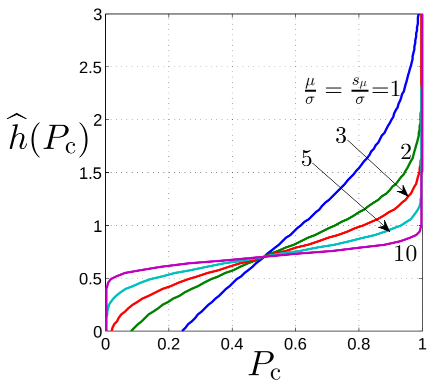


Figure 12: Robustness function, based on eq.129. (Transp.)

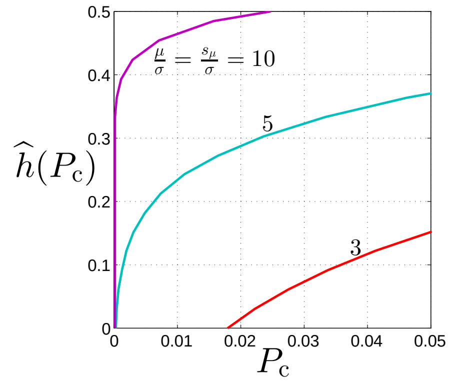


Figure 13: Robustness function, based on eq.129. (Transp.)

# 9 Uncertain Return, $i$, on Uncertain Constant Yearly Profit, $A$

§ **Background:** section 4.2, p. 8.

§ **Future worth** of constant profit, eq.(12), p.9:

- $A$ = profit at end of each period.
- $i$ = reinvest at profit rate $i$.
- $N$ = number of periods.
- The future worth, assuming that $i$ is the same in each period, is:

$$FW(A,i) = \sum_{k=0}^{N-1} (1+i)^{N-k} A = \frac{(1+i)^N - 1}{i} A \tag{130}$$

§ **Performance requirement:**

$$FW(A,i) \geq FW_\mathrm{c} \tag{131}$$

§ **Uncertainty:** $A$ and $i$ are both uncertain and constant, and we know $i \geq 0$ and $A \geq 0$ (or we can prevent $i < 0$ or $A \leq 0$, a loss).

Fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ A, i: \ A \geq 0, \ \left| \frac{A - \widetilde{A}}{s_A} \right| \leq h, \ i \geq 0, \ \left| \frac{i - \widetilde{i}}{s_i} \right| \leq h \right\}, \quad h \geq 0 \tag{132}$$

§ **Robustness:**

$$\widehat{h}(FW_\mathrm{c}) = \max \left\{ h: \ \left( \min_{A,i \in \mathcal{U}(h)} FW(A,i) \right) \geq FW_\mathrm{c} \right\} \tag{133}$$

§ **Evaluating the robustness:**

- Inner minimum:

$$m(h) = \min_{A,i \in \mathcal{U}(h)} FW(A,i) \tag{134}$$

- $m(h)$ vs $h$:
  - ◦ Decreasing function.
  - ◦ Recall eqs.(11) and (12), p.9:

$$F = \sum_{n=0}^{N-1} (1+i)^n A = \frac{(1+i)^N - 1}{i} A \tag{135}$$

- ∘ Inverse of $\widehat{h}(FW_{\rm c})$.
- ∘ From eqs.(130), (132) and (135), the inner minimum, $m(h)$, occurs at: $A = (\widetilde{A} - s_A h)^+$ and $i = \max(0, \widetilde{i} - s_i h) = (\widetilde{i} - s_i h)^+$.
- ∘ Thus:

$$m(h) = \left\{ \begin{array}{ll} \dfrac{(1+\widetilde{i} - s_i h)^N - 1}{\widetilde{i} - s_i h} (\widetilde{A} - s_A h)^+, & \text{for } h < \widetilde{i}/s_i \\ N(\widetilde{A} - s_A h)^+, & \text{for } h \geq \widetilde{i}/s_i \end{array} \right. \quad (136)$$

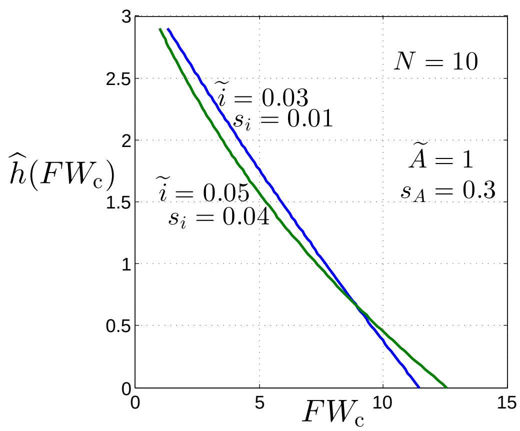


Figure 14: Robustness function, based on eq.136. (Transp.)

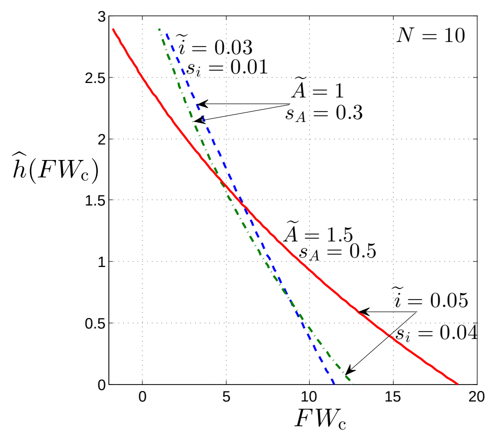


Figure 15: Robustness function, based on eq.136. (Transp.)

§ **Robustness functions,** fig. 14. $N = 10$, $\widetilde{A} = 1$, $s_A = 0.3$.

- Blue: $\widetilde{i} = 0.03$, $s_i = 0.01$. (Lower projected return; lower uncertainty.)
- Green: $\widetilde{i} = 0.05$, $s_i = 0.04$. (Higher projected return; higher uncertainty.)
- Similar, but mild preference reversal:
  Lower return ($\widetilde{i} = 0.03$) and lower uncertainty ($s_i = 0.01$) roughly equivalent to
  Higher return ($\widetilde{i} = 0.05$) and higher uncertainty ($s_i = 0.04$)

§ **Robustness functions,** fig. 15. $N = 10$.

- Blue: $\widetilde{i} = 0.03$, $s_i = 0.01$, $\widetilde{A} = 1$, $s_A = 0.3$. (Same a blue in fig. 14.)
- Green: $\widetilde{i} = 0.05$, $s_i = 0.04$, $\widetilde{A} = 1$, $s_A = 0.3$. (Same a green in fig. 14.)
- Red: $\widetilde{i} = 0.05$, $s_i = 0.04$, $\widetilde{A} = 1.5$, $s_A = 0.5$.
- Strong preference reversal between red and blue or green.

§ **Question:**

- The robustness curves in figs. 14, 15, p.34 are **decreasing** vs $FW_{\rm c}$.
- The robustness curves in figs. 12, 13, p.33 are **increasing** vs $P_{\rm c}$.
- Why the difference?
- Compare $\widehat{h}(P_{\rm c})$ in eq.(122), p.31, with $\widehat{h}(FW_{\rm c})$ in eq.(133), p.33:

$$\widehat{h}(P_{\rm c}) = \max \left\{ h : \left( \max_{p \in \mathcal{U}(h)} P_{\rm f}(p) \right) \leq P_{\rm c} \right\} \quad (137)$$

$$\widehat{h}(FW_{\rm c}) = \max \left\{ h : \left( \min_{A, i \in \mathcal{U}(h)} FW(A, i) \right) \geq FW_{\rm c} \right\} \quad (138)$$

# 10 Present and Future Worth Methods with Uncertainty

§ **Background:** section 5.

§ We will explore a few further examples and then address the question: are PW and FW preferences the same?

## 10.1 Example 5, p.17, Re-Visited

**Example 11** *Example 5, p.17, re-visited.*
§ Does the Present Worth method justify the following project,
**given uncertainty in revenue, cost and re-sale value?**

- $S$ = Initial cost of the project = \$10,000.
- $\widetilde{R}$ = estimated revenue at the end of $k$th period = \$5,310.
- $\widetilde{C}$ = estimated operating cost at the end of $k$th period = \$3,000.
- $\widetilde{M}$ = estimated re-sale value of equipment at end of project = \$2,000.
- $N$ = number of periods = 10.
- MARR = 10%, so $i = 0.1$.
- From eq.(49), p.17, the $PW$ is:

$$PW(R, C, M) = -S + \sum_{k=1}^{N} (1+i)^{-k} R_k - \sum_{k=1}^{N} (1+i)^{-k} C_k + (1+i)^{-N} M \tag{139}$$

- Fractional-error info-gap model for $R$, $C$ and $M$:

$$\mathcal{U}(h) = \left\{ R, C, M : \left| \frac{R_k - \widetilde{R}}{s_{R,k}} \right| \le h, \ \left| \frac{C_k - \widetilde{C}}{s_{C,k}} \right| \le h, \ k = 1, \ldots, N, \ \left| \frac{M - \widetilde{M}}{s_M} \right| \le h \right\}, \quad h \ge 0 \tag{140}$$

Consider expanding uncertainty envelopes for $R$ and $C$:

$$s_{x,k} = (1+\varepsilon)^{k-1} s_x, \quad x = R \text{ or } C \tag{141}$$

E.g., $\varepsilon = 0.1$. Note that $\varepsilon$ is like a discount rate on future uncertainty.

- Performance requirement:

$$PW(R, C, M) \ge PW_{\mathrm{c}} \tag{142}$$

- Robustness: greatest tolerable uncertainty:

$$\widehat{h}(PW_{\mathrm{c}}) = \max \left\{ h : \left( \min_{R,C,M \in \mathcal{U}(h)} PW(R, C, M) \right) \ge PW_{\mathrm{c}} \right\} \tag{143}$$

- The inner minimum, $m(h)$, occurs at **small** $R_k$ and $M$ and **large** $C_k$:

$$R_k = \widetilde{R} - s_{R,k} h = \widetilde{R} - (1+\varepsilon)^{k-1} s_R h \tag{144}$$

$$C_k = \widetilde{C} + s_{C,k} h = \widetilde{C} + (1+\varepsilon)^{k-1} s_C h \tag{145}$$

$$M = \widetilde{M} - s_M h \tag{146}$$

Thus $m(h)$ equals:

$$m(h) = -S + \sum_{k=1}^{N} (1+i)^{-k} \left[ \widetilde{R} - (1+\varepsilon)^{k-1} s_R h - \widetilde{C} - (1+\varepsilon)^{k-1} s_C h \right]$$

$$
\begin{aligned}
& +(1+i)^{-N}(\widetilde{M}-s_M h) && (147)\\
= & \underbrace{-S+(\widetilde{R}-\widetilde{C})\sum_{k=1}^{N}(1+i)^{-k} \; + \; (1+i)^{-N}\widetilde{M}}_{PW(\widetilde{R},\widetilde{C},\widetilde{M})} \\
& -\frac{s_R+s_c}{1+\varepsilon}h\underbrace{\sum_{k=1}^{N}\left(\frac{1+\varepsilon}{1+i}\right)^{k}}_{Q} - (1+i)^{-N}s_M h && (148)\\
= & PW(\widetilde{R},\widetilde{C},\widetilde{M}) - \left(\frac{s_R+s_c}{1+\varepsilon}Q+(1+i)^{-N}s_M\right)h && (149)
\end{aligned}
$$

Evaluate $Q$ with eq.(7), p.9, unless $\varepsilon = i$ in which case $Q = N$.

**Question:** $m(0) = PW(\widetilde{R},\widetilde{C},\widetilde{M})$. **Why?** What does this **mean?**

**Question:** $\mathrm{d}m(h)/\mathrm{d}h < 0$. **Why?** What does this **mean?**

- Equate $m(h)$ to $PW_\mathrm{c}$ and solve for $h$ to obtain the robustness:

$$
m(h) = PW_\mathrm{c} \quad \Longrightarrow \quad \boxed{\widehat{h}(PW_\mathrm{c}) = \frac{PW(\widetilde{R},\widetilde{C},\widetilde{M}) - PW_\mathrm{c}}{\frac{s_R+s_c}{1+\varepsilon}Q+(1+i)^{-N}s_M}} \tag{150}
$$

**See fig. 16, p.37**

- *Horizontal intercept* of the robustness curve. From eq.(52), p.17, we know:

$$
PW(\widetilde{R},\widetilde{C},\widetilde{M}) = -\$1.41 \tag{151}
$$

  - ∘ The project nominally almost breaks even.
  - ∘ Zeroing: no robustness at predicted outcome.

- *Slope* of the robustness curve is:

$$
\text{Slope} \;=-\left(\frac{s_R+s_c}{1+\varepsilon}Q+s_M\right)^{-1} \tag{152}
$$

Let $\varepsilon = i = 0.1$ so $Q = N = 10$. $s_R = 0.05\widetilde{R}$, $s_C = 0.03\widetilde{C}$, $s_M = 0.03\widetilde{M}$. Thus:

$$
\text{Slope} \;=-\left(\frac{0.05\times 5,310+0.03\times 3,000}{1.1}10+0.03\times 2,000\right)^{-1} = -1/3,291.82 \tag{153}
$$

Cost of robustness: $PW_\mathrm{c}$ must be **reduced** by \$3,291.82 in order to **increase** $\widehat{h}$ by 1 unit.

- **Decision making.** We need "several" units of robustness, say $\widehat{h}(PW_\mathrm{c}) \approx 3$ to 5. E.g.

$$
\widehat{h}(PW_\mathrm{c}) = 4 \quad \Longrightarrow \quad PW_\mathrm{c} = -\$13,168.69 \tag{154}
$$

Nominal $PW = -\$1.41$.

Reliable $PW = -\$13{,}168.69$.

Thus the incomes, $R_k$ and $M$, do not reliably cover the costs, $C_k$ and $S$. ■

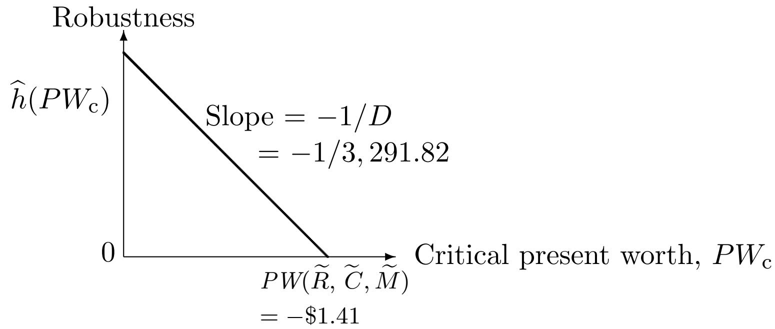


Figure 16: Robustness curve, eq.150, p.36, of example 11.

## 10.2 Example 7, p.19, Re-Visited

**Example 12** *Example 7, p.19, re-visited.*

§ Does the Present Worth method justify the following project,

**given uncertainty in revenue, operating and maintenance costs?**

- Project definition:
  - $P$ = initial investment = \$140,000.
  - $\widetilde{R}_k$ = estimated revenue at end of $k$th year = $\frac{2}{3}(45,000 + 5,000k)$.
  - $\widetilde{C}$= estimated operating cost paid at end of $k$th year = \$10,000.
  - $\widetilde{M}$ = estimated maintenance cost paid at end of $k$th year = \$1,800.
  - $T$ = tax and insurance paid at end of $k$th year = $0.02P = 2,800$.
  - $i = 0.15$ representing a MARR interest rate of 15%.
  - $N$ = 10 years.
- From eq.(60), p.19, the $PW$ is:

$$PW(R,C,M) \quad = \quad -P + \sum_{k=1}^{N}(R_k - C_k - M_k - T_k)(1+i)^{-k} \tag{155}$$

- Fractional-error info-gap model for $R$, $C$ and $M$:

$$\mathcal{U}(h) = \left\{ R,C,M : \ \left| \frac{R_k - \widetilde{R}_k}{s_{R,k}} \right| \le h, \ \left| \frac{C_k - \widetilde{C}}{s_{C,k}} \right| \le h, \ \left| \frac{M_k - \widetilde{M}}{s_{M,k}} \right| \le h, \ k = 1, \ldots, N \right\}, \quad h \ge 0 \tag{156}$$

Consider expanding uncertainty envelopes for $R$ and $C$:

$$s_{x,k} = (1+\varepsilon)^{k-1} s_x, \quad x = R,\ C,\ \text{or}\ M \tag{157}$$

E.g., $\varepsilon = 0.15$.

- Performance requirement:

$$PW(R,C,M) \ge PW_c \tag{158}$$

- Robustness: greatest tolerable uncertainty:

$$\widehat{h}(PW_c) = \max \left\{ h : \left( \min_{R,C,M \in \mathcal{U}(h)} PW(R,C,M) \right) \ge PW_c \right\} \tag{159}$$

- The inner minimum, $m(h)$, occurs at **small** $R_k$ and **large** $C_k$ and $M_k$:

$$\begin{aligned} R_k &= \widetilde{R}_k - s_{R,k}h = \widetilde{R}_k - (1+\varepsilon)^{k-1} s_R h && (160)\\ C_k &= \widetilde{C} + s_{C,k}h = \widetilde{C} + (1+\varepsilon)^{k-1} s_C h && (161)\\ M_k &= \widetilde{M} + s_{M,k}h = \widetilde{M} + (1+\varepsilon)^{k-1} s_M h && (162) \end{aligned}$$

Thus $m(h)$ equals:

$$
\begin{aligned}
m(h) &= -P \qquad (163)\\
&\quad + \sum_{k=1}^{N}(1+i)^{-k}\left[\widetilde{R}_k-(1+\varepsilon)^{k-1}s_R h-\widetilde{C}-(1+\varepsilon)^{k-1}s_C h-\widetilde{M}-(1+\varepsilon)^{k-1}s_M h-T_k\right]\\
&= \underbrace{-P+\sum_{k=1}^{N}(1+i)^{-k}\widetilde{R}_k \;-\; (\widetilde{C}+\widetilde{M}+T)\sum_{k=1}^{N}(1+i)^{-k}}_{PW(\widetilde{R},\widetilde{C},\widetilde{M})}\\
&\quad -\frac{s_R+s_C+s_M}{1+\varepsilon}h\underbrace{\sum_{k=1}^{N}\left(\frac{1+\varepsilon}{1+i}\right)^k}_{Q} \qquad (164)\\
&= PW(\widetilde{R},\widetilde{C},\widetilde{M})-\frac{s_R+s_C+s_M}{1+\varepsilon}Qh \qquad (165)
\end{aligned}
$$

Evaluate $Q$ with eq.(7), p.9, unless $\varepsilon = i$ in which case $Q = N$.

- Equate $m(h)$ to $PW_{\rm c}$ and solve for $h$ to obtain the robustness:

$$
m(h) = PW_{\rm c} \quad\Longrightarrow\quad \boxed{\widehat{h}(PW_{\rm c}) = \frac{PW(\widetilde{R},\widetilde{C},\widetilde{M})-PW_{\rm c}}{\frac{s_R+s_C+s_M}{1+\varepsilon}Q}} \qquad (166)
$$

Or zero if this is negative. See fig. 17, p.38.

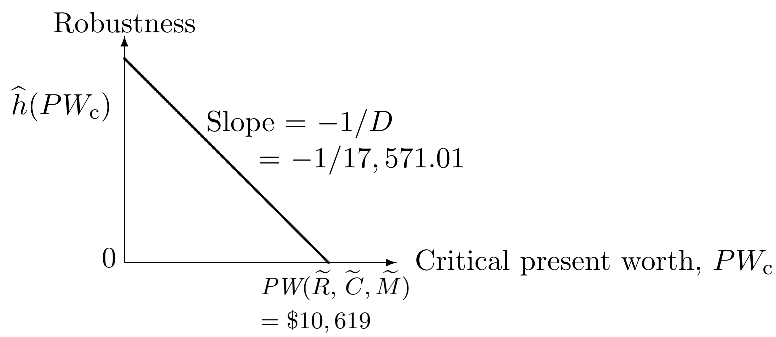


Figure 17: Robustness curve, eq.166, p.38, of example 12.

- *Horizontal intercept* of the robustness curve. From eq.(62), p.19, we know:

$$
PW(\widetilde{R},\widetilde{C},\widetilde{M}) = \$10,619. \qquad (167)
$$

  - ∘ The project nominally earns \$10,619.
  - ∘ Zeroing: no robustness at predicted outcome.

- *Slope* of the robustness curve is:

$$
\text{Slope} \;= -\left(\frac{s_R+s_C+s_M}{1+\varepsilon}Q\right)^{-1} \qquad (168)
$$

Let $\varepsilon = i = 0.15$ so $Q = N = 10$. $s_R = 0.05\widetilde{R}_1$, $s_C = 0.03\widetilde{C}$, $s_M = 0.03\widetilde{M}$. Thus:

$$
\text{Slope} \;= -\left(\frac{0.05\times(2/3)\times 50,000+0.03\times 10,000+0.03\times 1,800}{1.15}10\right)^{-1} = -1/17,571.01 \qquad (169)
$$

Cost of robustness: $PW_{\rm c}$ must be **reduced** by \$17,571.01 in order to **increase** $\widehat{h}$ by 1 unit.

- **Decision making.** We need "several" units of robustness, say $\widehat{h}(PW_{\rm c}) \approx 3$ to 5. E.g.

$$
\widehat{h}(PW_{\rm c}) = 4 \quad\Longrightarrow\quad PW_{\rm c} = -\$59,665.04 \qquad (170)
$$

Nominal $PW = +\$10{,}619$.

Reliable $PW = -\$59{,}665.04$.

Thus the incomes, $R_k$, do not cover the costs, $C_k$, $T_k$, $M_k$, and $P$.

- Compare examples 11 and 12, fig. 18, p.39.
  - Example 11: nominally worse but lower cost of robustness.
  - Example 12: nominally better but higher cost of robustness.
  - Preference reversal at $PW_{\text{c}} = -\$2,450$:

    Example 12 preferred for $PW_{\text{c}} > -\$2,450$, but robustness very low.

    Example 11 preferred for $PW_{\text{c}} < -\$2,450$.

■

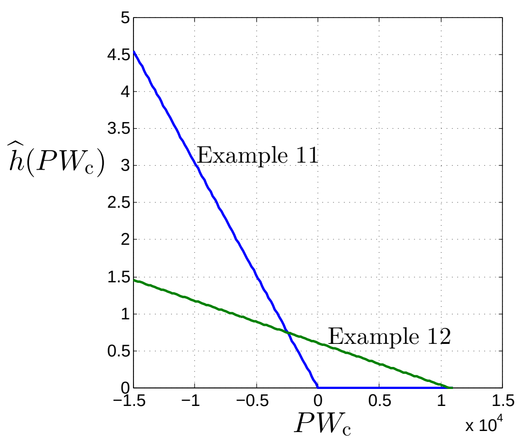


Figure 18: Robustness curves for examples 11 and 12, illustrating preference reversal. (Transp.)

## 10.3 Example 8, p.21, Re-Visited

**Example 13** *Example 8, p.21, re-visited (perhaps skip this example).*
§ Problem: Is the following investment worthwhile,

**given uncertainty in attaining the MARR in each period?**

- $F_0 = -\$25,000$ = cost of new equipment.
- $F = \$8,000$ net revenue (after operating cost), $k = 1, \ldots, 5$.
- $N = 5$ = planning horizon.
- $M = \$5,000$ = market value of equipment at end of planning horizon.
- $\widetilde{i} = 0.2 = 20\%$ is the **anticipated** MARR.
- From eq.(69), p.21, the **anticipated** $FW$ is:

$$\widetilde{FW} = M + \sum_{k=0}^{N} (1+\widetilde{i})^{N-k} F_k \tag{171}$$

where $F_k = F$ for $k > 0$.

- We desire $\widetilde{i} = 0.2$, but we may not attain this high rate of return each period.
- Define a new discount rate in the $k$th period as:

$$\beta_k = (1+i)^{N-k}, \quad k = 0, \ldots, N \tag{172}$$

where $i$ may vary from period to period.

The anticipated value is:

$$\widetilde{\beta}_k = (1+\widetilde{i})^{N-k}, \quad k = 0, \ldots, N \tag{173}$$

- Thus the anticipated and actual $FW$'s are:

$$\widetilde{FW} = M + \sum_{k=0}^{N} \widetilde{\beta}_k F_k \tag{174}$$

$$FW = M + \sum_{k=0}^{N} \beta_k F_k \tag{175}$$

- A fractional-error info-gap model for the discount rates, treating the uncertainty separately in each period, is:

$$\mathcal{U}(h) = \left\{ \beta : \ \beta_k \ge 0, \ \left| \frac{\beta_k - \widetilde{\beta}_k}{s_k} \right| \le h, \ k = 0, \ldots, N \right\}, \quad h \ge 0 \tag{176}$$

◦ The uncertainty weights, $s_k$, may increase over time.
◦ $\beta_k \ge 0$ because $i \ge -1$.
◦ **Treating the uncertainty separately** in each period is a strong approximation, and **really not justified.** From eq.(26), p.13, we see that $\beta_k$ is related to $\beta_{k-1}$. The full analysis is much more complicated.

- Performance requirement:

$$FW(\beta) \ge FW_{\mathrm{c}} \tag{177}$$

- Robustness:

$$\widehat{h}(FW_{\mathrm{c}}) = \max \left\{ h : \ \left( \min_{\beta \in \mathcal{U}(h)} FW(\beta) \right) \ge FW_{\mathrm{c}} \right\} \tag{178}$$

- Evaluate the inner minimum, $m(h)$: inverse of the robustness. Occurs at:

$$\beta_0 = \widetilde{\beta}_0 + s_0 h \text{ because } F_0 < 0, \quad \beta_k = \max[0, \ \widetilde{\beta}_k - s_k h], \ k = 1, \ldots, N \tag{179}$$

So:

$$m(h) \;=\; M + (\widetilde{\beta}_0 + s_0 h)F_0 + F\sum_{k=1}^{N}\max[0,\ \widetilde{\beta}_k - s_k h] \qquad (180)$$

Define:

$$h_1 = \min_{1\le k\le N} \frac{\widetilde{\beta}_k}{s_k} \qquad (181)$$

For $h \le h_1$ we can write eq.(180) as:

$$\begin{aligned} m(h) &= \underbrace{M + \sum_{k=0}^{N}\widetilde{\beta}_k F_k}_{\widetilde{FW}} \;-\; h\;\underbrace{\left(-s_0F_0 + F\sum_{k=1}^{N} s_k\right)}_{FW^\star} && (182)\\ &= \widetilde{FW} - hFW^\star && (183)\end{aligned}$$

Note that $FW^\star > 0$.

- Equate eq.(183) to $FW_{\rm c}$ and solve for $h$ to obtain **part** of the robustness curve:

$$\widehat{h}(FW_{\rm c}) = \frac{\widetilde{FW} - FW_{\rm c}}{FW^\star}, \quad \widetilde{FW} - h_1FW^\star \le FW_{\rm c} \le \widetilde{FW} \qquad (184)$$

- Note possibility of crossing robustness curves and preference reversal.
- For $h > h_1$, successive terms in eq.(180) drop out and the slope of the robustness curve changes.
- **Question:** How can we plot the **entire** robustness curve, without the constraint $h \le h_1$?

■

### 10.4 Info-Gap on $A$: Are $\boldsymbol{PW}$ and $\boldsymbol{FW}$ Robust Preferences the Same?

§ Continue example of section 8.1, p.27 (constant yearly profit), where the $FW$, eq.(93) p.27, is:

$$FW = \frac{(1+i)^N - 1}{i} A \tag{185}$$

and the uncertainty is only in $A$, eq.(94) p.27, is:

$$\mathcal{U}(h) = \left\{ A : \left| \frac{A - \widetilde{A}}{s_A} \right| \le h \right\}, \quad h \ge 0 \tag{186}$$

and the performance requirement, eq.(95) p.27, is:

$$FW(A) \ge FW_{\rm c} \tag{187}$$

§ $PW$ and $FW$ are related by eq.(66), p.20:

$$PW(A) = (1+i)^{-N} FW(A) \tag{188}$$

§ Thus, from eqs.(187) and (188), the performance requirement for $PW$ is:

$$PW(A) \ge PW_{\rm c} \tag{189}$$

where:

$$PW_{\rm c} = (1+i)^{-N} FW_{\rm c} \tag{190}$$

§ The robustness for the $FW$ criterion is $\widehat{h}_{fw}(FW_{\rm c})$, eq.(96) p.27, is:

$$\widehat{h}_{fw}(FW_{\rm c}) = \max \left\{ h : \left( \min_{A \in \mathcal{U}(h)} FW(A) \right) \ge FW_{\rm c} \right\} \tag{191}$$

§ The robustness for the $PW$ criterion is $\widehat{h}_{pw}(PW_{\rm c})$, is defined analogously:

$$\widehat{h}_{pw}(PW_{\rm c}) = \max \left\{ h : \left( \min_{A \in \mathcal{U}(h)} PW(A) \right) \ge PW_{\rm c} \right\} \tag{192}$$

Employing eqs.(188) and (190) we obtain:

$$\begin{aligned} \widehat{h}_{pw}(PW_{\rm c}) &= \max \left\{ h : \left( \min_{A \in \mathcal{U}(h)} (1+i)^{-N} FW(A) \right) \ge (1+i)^{-N} FW_{\rm c} \right\} && (193) \\ &= \widehat{h}_{fw}(FW_{\rm c}) && (194) \end{aligned}$$

because $(1+i)^{-N}$ cancels out in eq.(193). The values differ, but the robustnesses are equal!

§ Consider two different configurations, $k = 1, 2$, whose robustness functions are $\widehat{h}_{pw,k}(PW_{\rm c})$ and $\widehat{h}_{fw,k}(FW_{\rm c})$.

- From eq.(194) we see that:

$$\widehat{h}_{pw,1}(PW_{\rm c}) > \widehat{h}_{pw,2}(PW_{\rm c}) \quad \text{if and only if} \quad \widehat{h}_{fw,1}(FW_{\rm c}) > \widehat{h}_{fw,2}(FW_{\rm c}) \tag{195}$$

- Thus $FW$ and $PW$ robust preferences between the configurations are the same **when $A$ is the only uncertainty**.

### 10.5 Info-Gap on $i$: Are $PW$ and $FW$ Robust Preferences the Same?

§ Continue example of section 8.1, p.27 (constant yearly profit), where the $FW$, eq.(93) p.27, is:

$$FW = \frac{(1+i)^N - 1}{i} A \tag{196}$$

where $i$ is constant but uncertain:

$$\mathcal{U}(h) = \left\{ i: \ i \geq -1, \ \left| \frac{i - \tilde{i}}{s_i} \right| \leq h \right\}, \quad h \geq 0 \tag{197}$$

and the performance requirement, eq.(95) p.27, is:

$$FW(i) \geq FW_{\mathrm{c}} \tag{198}$$

§ $PW$ and $FW$ are related by eq.(66), p.20:

$$PW(i) = (1+i)^{-N} FW(i) \tag{199}$$

§ Thus, from eqs.(198) and (199), the performance requirement for $PW$ is

$$PW(i) \geq PW_{\mathrm{c}} \tag{200}$$

where:

$$PW_{\mathrm{c}} = (1+i)^{-N} FW_{\mathrm{c}} \tag{201}$$

However, because $i$ is uncertain we will write the performance requirement as:

$$PW(i) - (1+i)^{-N} FW_{\mathrm{c}} \geq 0 \tag{202}$$

§ The robustness for the $FW$ criterion is:

$$\widehat{h}_{fw}(FW_{\mathrm{c}}) = \max \left\{ h: \ \left( \min_{i \in \mathcal{U}(h)} FW(i) \right) \geq FW_{\mathrm{c}} \right\} \tag{203}$$

We re-write this as:

$$\widehat{h}_{fw}(FW_{\mathrm{c}}) = \max \left\{ h: \ \left( \min_{i \in \mathcal{U}(h)} (FW(i) - FW_{\mathrm{c}}) \right) \geq 0 \right\} \tag{204}$$

Let $m_{fw}(h)$ denote the inner minimum, which is the inverse of $\widehat{h}_{fw}(FW_{\mathrm{c}})$.

§ The robustness for the $PW$ criterion is:

$$\widehat{h}_{pw}(FW_{\mathrm{c}}) = \max \left\{ h: \ \left( \min_{i \in \mathcal{U}(h)} \left( PW(i) - (1+i)^{-N} FW_{\mathrm{c}} \right) \right) \geq 0 \right\} \tag{205}$$

$$= \max \left\{ h: \ \left( \min_{i \in \mathcal{U}(h)} (1+i)^{-N} (FW(i) - FW_{\mathrm{c}}) \right) \geq 0 \right\} \tag{206}$$

- Let $m_{pw}(h)$ denote the inner minimum, which is the inverse of $\widehat{h}_{pw}(FW_{\mathrm{c}})$.
- Because $(1+i)^{-N} > 0$, we **can** conclude that:

$$m_{fw}(h) \geq 0 \quad \text{if and only if} \quad m_{pw}(h) \geq 0 \tag{207}$$

- Define $\mathcal{H}_{fw}$ as the set of $h$ values in eq.(204) whose maximum is $\widehat{h}_{fw}(FW_{\rm c})$.
- Define $\mathcal{H}_{pw}$ as the set of $h$ values in eq.(206) whose maximum is $\widehat{h}_{pw}(FW_{\rm c})$.
- Eq.(207) implies that:

$$h \in \mathcal{H}_{fw} \quad \text{if and only if} \quad h \in \mathcal{H}_{pw} \tag{208}$$

which implies that:

$$\max \mathcal{H}_{fw} = \max \mathcal{H}_{pw} \tag{209}$$

which implies that:

$$\widehat{h}_{fw}(FW_{\rm c}) = \widehat{h}_{pw}(FW_{\rm c}) \tag{210}$$

§ Thus $FW$ and $PW$ robust preferences between the configurations are the same **when $i$ is the only uncertainty**.

§ A different proof of eq.(210) is (we might skip this proof):

- From the definition of $\widehat{h}_{fw}$, eq.(204), we conclude that:

$$m_{fw}(\widehat{h}_{fw}) \geq 0 \tag{211}$$

and this implies, from eq.(207), that:

$$m_{pw}(\widehat{h}_{fw}) \geq 0 \tag{212}$$

From this and from the definition of $\widehat{h}_{pw}$, eq.(206), we conclude that:

$$\widehat{h}_{pw} \geq \widehat{h}_{fw} \tag{213}$$

- Likewise, from the definition of $\widehat{h}_{pw}$, eq.(206), we conclude that:

$$m_{pw}(\widehat{h}_{pw}) \geq 0 \tag{214}$$

and this implies, from eq.(207), that:

$$m_{fw}(\widehat{h}_{pw}) \geq 0 \tag{215}$$

From this and from the definition of $\widehat{h}_{fw}$, eq.(204), we conclude that:

$$\widehat{h}_{fw} \geq \widehat{h}_{pw} \tag{216}$$

- Combining eqs.(213) and (216) we find:

$$\widehat{h}_{fw}(FW_{\rm c}) = \widehat{h}_{pw}(FW_{\rm c}) \tag{217}$$

- QED.

# 11 Strategic Uncertainty

§ **Strategic interaction:**

- Competition between protagonists.
- Willful goal-oriented behavior.
- Knowledge of each other.
- Potential for deliberate interference or deception.

## 11.1 Preliminary (Non-Strategic) Example: 1 Allocation

§ **1 allocation:**

- Allocate positive quantity $F_0$ at time step $t = 0$.
- This results in future income $F_1$ at time step $t = 1$:

$$F_1 = bF_0 \tag{218}$$

  - Eq.(218) is the **system model**.
  - $b$ is the "**budget effectiveness**".
  - $\widetilde{b}$ is the estimated value of $b$, where $b$ is **uncertain**.

§ **A fractional-error info-gap model for uncertainty** in $b$:

$$\mathcal{U}(h) = \left\{ b : \left| \frac{b - \widetilde{b}}{s_b} \right| \le h \right\}, \quad h \ge 0 \tag{219}$$

§ **Performance requirement:**

$$F_1 \ge F_{1c} \tag{220}$$

§ **Definition of robustness** of allocation $F_0$:

$$\widehat{h}(F_{1c}, F_0) = \max \left\{ h : \left( \min_{b \in \mathcal{U}(h)} F_1 \right) \ge F_{1c} \right\} \tag{221}$$

§ **Evaluation of robustness:**

- $m(h)$ denotes inner minimum in eq.(221).
- $m(h)$ is the inverse of $\widehat{h}(F_{1c}, F_0)$ thought of as a function of $F_{1c}$.
- $F_0 > 0$, so $m(h)$ occurs at $b = \widetilde{b} - s_b h$:

$$m(h) = (\widetilde{b} - s_b h)F_0 \ge F_{1c} \quad \Longrightarrow \quad \boxed{\widehat{h}(F_{1c}, F_0) = \frac{\widetilde{b}F_0 - F_{1c}}{F_0 s_b}} \tag{222}$$

  or zero if this is negative. See fig. 19, p.46.
- **Zeroing:** no robustness when $F_{1c} = F_1(\widetilde{b})$.
- **Trade off:** robustness increases as requirement, $F_{1c}$, becomes less demanding (smaller).
- **Preference reversal and its dilemma:**
  - Consider two options:

$$(\widetilde{b}F_0)_1 \quad < \quad (\widetilde{b}F_0)_2 \quad \text{Option 2 purportedly better} \tag{223}$$

$$\left( \frac{\widetilde{b}}{s_b} \right)_1 \quad > \quad \left( \frac{\widetilde{b}}{s_b} \right)_2 \quad \text{Option 2 more uncertain} \tag{224}$$

  - Eq.(223) compares the horizontal intercepts at $\widehat{h} = 0$.
  - Eq.(224) compares the vertical intercepts at $F_{1c} = 0$.
  - Robustness curves cross one another: potential preference reversal; fig. 20, p.46.

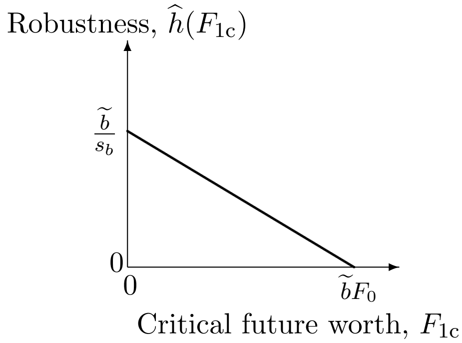


Figure 19: Robustness curve, eq.(222).

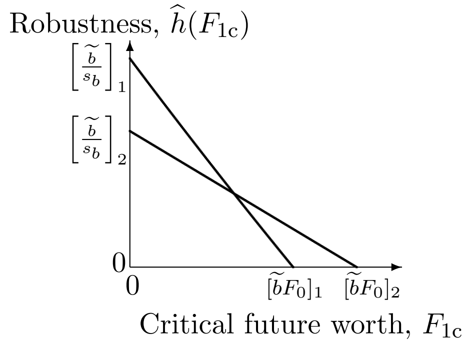


Figure 20: Preference reversal, eqs.(223) and (224).

## 11.2 1 Allocation with Strategic Uncertainty

§ Continuation of example in section 11.1.

§ **Strategic interaction:**
- Competition between protagonists.
- Willful goal-oriented behavior.
- Knowledge of each other.
- Potential for deliberate interference or deception.

§ **1 allocation:**
- Invest positive quantity $F_0$ at time step $t = 0$.
- This results in future income $F_1$ at time step $t = 1$:

$$F_1 = bF_0 \tag{225}$$

  - ○ Eq.(225) is the **system model**.
  - ○ $b$ is the "**budget effectiveness**" which is uncertain.

§ **Budget effectiveness:**
- "Our" budget effectiveness is influenced by a choice, $c$, made by "them":

$$b(c) = \widetilde{b}_0 - \alpha c \tag{226}$$

where $\alpha > 0$. Suppose that **only** $c$ **is uncertain.**
- $\alpha$ is the "aggressiveness" of their choice.

§ **A fractional-error info-gap model for uncertainty** in $c$:

$$\mathcal{U}(h) = \left\{ c : \left| \frac{c - \widetilde{c}}{s_c} \right| \le h \right\}, \quad h \ge 0 \tag{227}$$

§ **Performance requirement:**

$$F_1 \ge F_{1c} \tag{228}$$

§ **Definition of robustness** of allocation $F_0$:

$$\widehat{h}(F_{1c}, F_0) = \max \left\{ h : \left( \min_{c \in \mathcal{U}(h)} F_1 \right) \ge F_{1c} \right\} \tag{229}$$

§ **Evaluation of robustness:**

- $m(h)$ denotes inner minimum in eq.(229): the inverse of $\widehat{h}(F_{1c}, F_0)$ as function of $F_{1c}$.
- $F_0 > 0$ and $\alpha > 0$, so $m(h)$ occurs at $c = \widetilde{c} + s_c h$:

$$m(h) = \left[\widetilde{b}_0 - \alpha(\widetilde{c} + s_c h)\right] F_0 \geq F_{1c} \quad \Longrightarrow \tag{230}$$

$$\widehat{h}(F_{1c}, F_0) = \frac{(\widetilde{b}_0 - \alpha\widetilde{c})F_0 - F_{1c}}{\alpha s_c F_0} \tag{231}$$

$$= \frac{F_1(\widetilde{c}) - F_{1c}}{\alpha s_c F_0} \tag{232}$$

or zero if this is negative.

- **Zeroing** (fig. 21): no robustness when $F_{1c} = F_1(\widetilde{c})$.
- **Trade off** (fig. 21): robustness increases as requirement, $F_{1c}$, becomes less demanding (smaller).

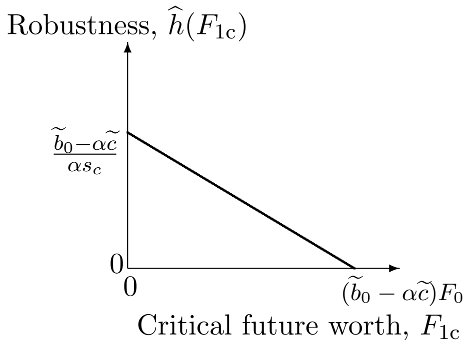


Figure 21: Robustness curve, eq.(231).

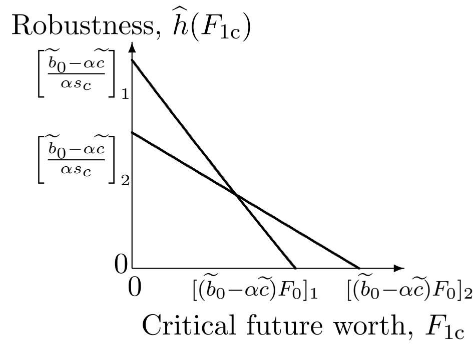


Figure 22: Robustness curve, eq.(231).

§ **Preference reversal** (fig. 22):

- Consider two options:

$$[(\widetilde{b}_0 - \alpha\widetilde{c})F_0]_1 < [(\widetilde{b}_0 - \alpha\widetilde{c})F_0]_2 \quad \text{Option 2 purportedly better} \tag{233}$$

$$\left(\frac{\widetilde{b}_0 - \alpha\widetilde{c}}{\alpha s_c}\right)_1 > \left(\frac{\widetilde{b}_0 - \alpha\widetilde{c}}{\alpha s_c}\right)_2 \quad \text{Option 2 more uncertain} \tag{234}$$

- A possible interpretation. “They” in option 2 are:
  - ◦ Purportedly less aggressive: $\alpha_2 < \alpha_1 \quad \Longrightarrow \quad$ eq.(233).
  - ◦ Much less well known to “us”: $s_{c2} \gg s_{c1} \quad \Longrightarrow \quad$ eq.(234).
- Robustness curves cross one another: potential for preference reversal.

## 11.3 2 Allocations with Strategic Uncertainty

§ **System model.** 2 non-negative allocations, $F_{01}$ and $F_{02}$, at time step 0:

$$F_{11} = b_1 F_{01} \tag{235}$$

$$F_{12} = b_2 F_{02} \tag{236}$$

§ **Budget constraint:**

$$F_{01} + F_{02} = F_{\max}, \quad F_{0k} \geq 0, \quad k = 1,\ 2 \tag{237}$$

§ **Performance requirement:**

$$F_{11} + F_{12} \geq F_{1c} \tag{238}$$

§ **Budget effectiveness:**

- "Our" budget effectiveness is influenced by choices, $c_k$, made by "them":

$$b_k(c) = \widetilde{b}_{0k} - \alpha_k c_k, \quad k = 1,\ 2 \tag{239}$$

where $\alpha_k > 0$. Suppose that **only** $c_1$ **and** $c_2$ **are uncertain,** with estimates $\widetilde{c}_1$ and $\widetilde{c}_2$.

§ **Purported optimal allocation,** assuming **no uncertainty:**

- Aim to maximize $F_{11} + F_{12}$.
- Put all funds on better anticipated investment:

$$\text{If: } b_k(\widetilde{c}_k) > b_j(\widetilde{c}_j) \quad \text{then: } F_{0k} = F_{\max} \text{ and } F_{0j} = 0 \tag{240}$$

§ **A fractional-error info-gap model for uncertainty** in $c$:

$$\mathcal{U}(h) = \left\{ c : \left| \frac{c_k - \widetilde{c}_k}{s_k} \right| \leq h, \quad k = 1,\ 2 \right\}, \quad h \geq 0 \tag{241}$$

§ **Definition of robustness** of allocation vector $F_0$:

$$\widehat{h}(F_{1c}, F_0) = \max \left\{ h : \left( \min_{c \in \mathcal{U}(h)} (F_{11} + F_{12}) \right) \geq F_{1c} \right\} \tag{242}$$

§ **Evaluation of robustness:**

- $m(h)$ denotes inner minimum in eq.(242): the inverse of $\widehat{h}(F_{1c}, F_0)$ as function of $F_{1c}$.
- $F_{0k} \geq 0$ and $\alpha_k > 0$, so $m(h)$ occurs at $c_k = \widetilde{c}_k + s_k h$, $k = 1,\ 2$:

$$m(h) = \sum_{k=1}^{2} \left[ \widetilde{b}_{0k} - \alpha_k(\widetilde{c}_k + s_k h) \right] F_{0k} \tag{243}$$

$$= \underbrace{\sum_{k=1}^{2} \left[ \widetilde{b}_{0k} - \alpha_k \widetilde{c}_k) \right] F_{0k}}_{F_1(\widetilde{c}) = \widetilde{b}^T F_0} - h \underbrace{\sum_{k=1}^{2} \alpha_k s_k F_{0k}}_{\sigma^T F_0} \tag{244}$$

$$= F_1(\widetilde{c}) - h \sigma^T F_0 \tag{245}$$

which defines the vectors $\widetilde{b}$, $F_0$ and $\sigma$.

• Equate $m(h)$ to $F_{1c}$ and solve for $h$ to obtain the robustness:

$$m(h) = F_{1c} \quad \Longrightarrow \quad \widehat{h}(F_{1c}, F_0) \; = \; \frac{F_1(\widetilde{c}) - F_{1c}}{\sigma^T F_0} \tag{246}$$

$$= \; \frac{\widetilde{b}^T F_0 - F_{1c}}{\sigma^T F_0} \tag{247}$$

or zero if this is negative.

• **Zeroing** (fig. 23): no robustness when $F_{1c} = F_1(\widetilde{c})$.

• **Trade off** (fig. 23): robustness increases as requirement, $F_{1c}$, becomes less demanding (smaller).

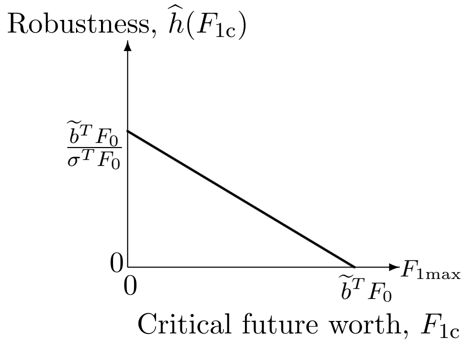


Figure 23: Robustness curve, eq.(247).

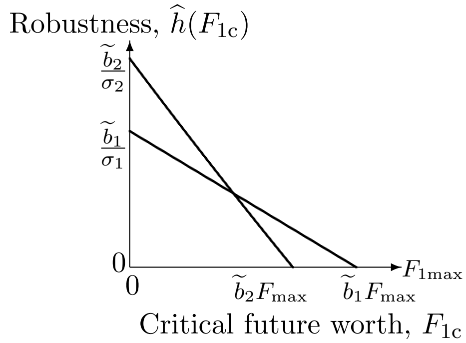


Figure 24: Robustness curves for extreme allocations eqs.(248), (249).

§ **Two extreme allocations,** the purported best and worst allocations:

• Suppose $b_1(\widetilde{c}_1) > b_2(\widetilde{c}_2)$, so:

◦ $F_{01} = F_{\max}$, $F_{02} = 0$ is **purportedly best**:

$$\widehat{h}(F_{01} = F_{\max}) = \frac{b_1(\widetilde{c}_1)F_{\max} - F_{1c}}{\sigma_1 F_{\max}} \tag{248}$$

◦ $F_{01} = 0$, $F_{02} = F_{\max}$ is **purportedly worst**:

$$\widehat{h}(F_{02} = F_{\max}) = \frac{b_2(\widetilde{c}_2)F_{\max} - F_{1c}}{\sigma_2 F_{\max}} \tag{249}$$

• Also suppose: $\dfrac{\widetilde{b}_1}{\sigma_1} < \dfrac{\widetilde{b}_2}{\sigma_2}$ so first option is **more uncertain**.

• **Preference reversal,** fig. 24:

The purported best allocation is **less robust** than
the purported worst allocation for some values of $F_{1c}$.

• The most robust option is still allocation to only one asset, but not necessarily to the nominally optimal asset.

## 11.4 Asymmetric Information and Strategic Uncertainty: Employment Offer

§ **Employer's problem:**

- Employer wants to hire an employee.
- Employer must offer a salary to the employee, who can refuse the offer. No negotiation.
- Employer does not know either the true economic value, or the refusal price, of the employee.

§ **Employer's NPV (Net Present Value):**

- $C$ = pay at end of each of $N$ periods offered to employee.
- $A$ = uncertain income, at end of each of $N$ periods, to employer from employee's work.
- Employer's NPV, adapting eq.(45), p.17:

$$PW = \sum_{k=1}^{N}(1+i)^{-k}(A-C) \tag{250}$$

$$= \underbrace{\frac{1-(1+i)^{-N}}{i}}_{\mathcal{I}}(A-C) \tag{251}$$

where eq.(251) employs eq.(9), p.9.

- The employer's $PW$ requirement:

$$PW \geq PW_{\mathrm{c}} \tag{252}$$

§ **Uncertainty about $A$:**

- **Asymmetric information:**
  - The employee knows things about himself that the employer does not know.
  - The prospective employee states that his work will bring in $\widetilde{A}$ each period.
  - The employee thinks this is an over-estimate but does not know by how much.
  - The employer adopts an asymmetric fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ A: \ 0 \leq \frac{\widetilde{A}-A}{\widetilde{A}} \leq h \right\}, \quad h \geq 0 \tag{253}$$

Note **asymmetrical uncertainty** resulting from **asymmetrical information.**

§ **Employer's offered contract and employee's potential refusal:**

- The employer will offer to pay the employee $C$ per period.
- The employee will refuse if this is less that his refusal cost, $C_{\mathrm{r}}$.
- The employer wants to choose $C$ so probability of refusal is less than $\varepsilon$, where $\varepsilon \leq \frac{1}{2}$.
- The employer doesn't know employee's value of $C_{\mathrm{r}}$ and only has a guess of pdf of $C_{\mathrm{r}}$.
- Once again: **asymmetric information.**
- The employer's estimate of the pdf of $C_{\mathrm{r}}$ is $\widetilde{p}(C_{\mathrm{r}})$, which is $\mathcal{N}(\mu, \sigma^2)$.
- Employer chooses $\mu < \widetilde{A}$ to reflect asymmetrical information.
- The employer's info-gap model for uncertainty in this pdf is:

$$\mathcal{V}(h) = \left\{ p(C_{\mathrm{r}}): \ p(C_{\mathrm{r}}) \geq 0, \ \int_{-\infty}^{\infty} p(C_{\mathrm{r}})\,\mathrm{d}C_{\mathrm{r}} = 1, \ \left| \frac{p(C_{\mathrm{r}}) - \widetilde{p}(C_{\mathrm{r}})}{\widetilde{p}(C_{\mathrm{r}})} \right| \leq h \right\}, \quad h \geq 0 \tag{254}$$

- The probability of refusal by the employee, of the offered value of $C$, is (see fig. 25, p.51):

$$P_{\mathrm{ref}}(C|p) = \mathrm{Prob}(C_{\mathrm{r}} \geq C) = \int_{C}^{\infty} p(C_{\mathrm{r}})\,\mathrm{d}C_{\mathrm{r}} \tag{255}$$

- The employer's requirement regarding employee refusal, where $\varepsilon \leq \frac{1}{2}$, is:

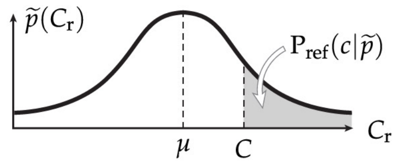


Figure 25: Probability of refusal by the employee, eq.(255).

$$P_{\text{ref}}(C|p) \le \varepsilon \tag{256}$$

§ **Definition of the robustness:**

• Overall robustness:

$$\widehat{h}(C, PW_{\text{c}}, \varepsilon) = \max\left\{h:\ \left(\min_{A\in\mathcal{U}(h)} PW(C,A)\right) \ge PW_{\text{c}},\quad \left(\max_{p\in\mathcal{V}(h)} P_{\text{ref}}(C|p)\right) \le \varepsilon\right\} \tag{257}$$

• This can be expressed in terms of two **sub-robustnesses.**

• Robustness of $PW$:

$$\widehat{h}_{\text{pw}}(C, PW_{\text{c}}) = \max\left\{h:\ \left(\min_{A\in\mathcal{U}(h)} PW(C,A)\right) \ge PW_{\text{c}}\right\} \tag{258}$$

• Robustness of employee refusal:

$$\widehat{h}_{\text{ref}}(C, \varepsilon) = \max\left\{h:\ \left(\max_{p\in\mathcal{V}(h)} P_{\text{ref}}(C|p)\right) \le \varepsilon\right\} \tag{259}$$

• The overall robustness can be expressed:

$$\widehat{h}(C, PW_{\text{c}}, \varepsilon) = \min\left[\widehat{h}_{\text{pw}}(C, PW_{\text{c}}),\ \widehat{h}_{\text{ref}}(C,\varepsilon)\right] \tag{260}$$

• **Why minimum** in eq.(260)?

• Both performance requirements, eqs.(252) and (256), must be satisfied, so the overall robustness is the lower of the two sub-robustnesses.

§ **Evaluating** $\widehat{h}_{\text{pw}}(C, PW_{\text{c}})$:

• Let $m_{\text{pw}}(h)$ denote the inner minimum in eq.(258).

• $m_{\text{pw}}(h)$ is the inverse of $\widehat{h}_{\text{pw}}(C, PW_{\text{c}})$ thought of as a function of $PW_{\text{c}}$.

• Eq.(251): $PW = (A - C)\mathcal{I}$. Thus $m_{\text{pw}}(h)$ occurs for $A = (1-h)\widetilde{A}$ ($\mathcal{I}$ is defined in eq.(251), p.50):

$$m_{\text{pw}}(h) = \left[(1-h)\widetilde{A} - C\right]\mathcal{I} \ge PW_{\text{c}} \quad \Longrightarrow \tag{261}$$

$$\widehat{h}_{\text{pw}}(C, PW_{\text{c}}) = \frac{(\widetilde{A} - C)\mathcal{I} - PW_{\text{c}}}{\widetilde{A}\mathcal{I}} \tag{262}$$

$$= \boxed{\frac{PW(\widetilde{A}) - PW_{\text{c}}}{\widetilde{A}\mathcal{I}}} \tag{263}$$

or zero if this is negative.

§ **Evaluating** $\widehat{h}_{\text{ref}}(C, \varepsilon)$:

• Let $m_{\text{ref}}(h)$ denote the inner maximum in eq.(259).

• $m_{\text{ref}}(h)$ is the inverse of $\widehat{h}_{\text{ref}}(C,\varepsilon)$ thought of as a function of $\varepsilon$.

• Recall: $\varepsilon \le \frac{1}{2}$.

• Thus, we must choose $C$ to be **no less than median** of $\widetilde{p}(C_{\rm r})$ because we require (see fig. 26, p.52):

$$P_{\rm ref}(C|\widetilde{p}) = \int_C^\infty \widetilde{p}(C_{\rm r})\,{\rm d}C_{\rm r} \le \varepsilon \le \frac{1}{2} \tag{264}$$

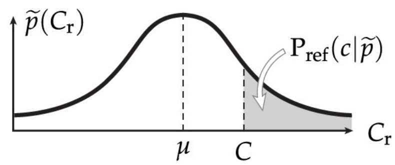


Figure 26: Probability of refusal by the employee, eq.(255).

• Eq.(255): $P_{\rm ref}(C|p) = {\rm Prob}(C_{\rm r} \ge C) = \int_C^\infty p(C_{\rm r})\,{\rm d}C_{\rm r}$. For $h \le 1$, $m_{\rm ref}(h)$ occurs for:

$$p(C_{\rm r}) = \begin{cases} (1+h)\widetilde{p}(C_{\rm r}), & C_{\rm r} \ge C \\ (1-h)\widetilde{p}(C_{\rm r}), & \text{for part of } C_{\rm r} < C \text{ to normalize } p(C_{\rm r}) \\ \widetilde{p}(C_{\rm r}), & \text{for remainder of } C_{\rm r} < C \end{cases} \tag{265}$$

**Why don't we care** what "part of $C_{\rm r} < C$" in the middle line of eq.(265)?

• Thus, for $h \le 1$:

$$m_{\rm ref}(h) = \int_C^\infty (1+h)\widetilde{p}(C_{\rm r})\,{\rm d}C_{\rm r} \tag{266}$$

$$= (1+h){\rm Prob}(C_{\rm r} \ge C|\widetilde{p}) = (1+h){\rm Prob}\left(\frac{C_{\rm r}-\mu}{\sigma} \ge \frac{C-\mu}{\sigma}\bigg|\,\widetilde{p}\right) \tag{267}$$

$$= (1+h)\left[1 - \Phi\left(\frac{C-\mu}{\sigma}\right)\right] \le \varepsilon \qquad \left(\text{because } \frac{C_{\rm r}-\mu}{\sigma} \sim \mathcal{N}(0,1)\right) \tag{268}$$

$$\Longrightarrow \boxed{\widehat{h}_{\rm ref}(C,\varepsilon) = \frac{\varepsilon}{1-\Phi\left(\frac{C-\mu}{\sigma}\right)} - 1}$$

$$\text{for } 1 - \Phi\left(\frac{C-\mu}{\sigma}\right) \le \varepsilon \le 2\left[1 - \Phi\left(\frac{C-\mu}{\sigma}\right)\right] \tag{269}$$

◦ Note that $\widehat{h}_{\rm ref}(C,\varepsilon) \le 1$ for the $\varepsilon$-range indicated, so assumption that $h \le 1$ is satisfied.
◦ We have not derived $\widehat{h}_{\rm ref}$ for $\varepsilon$ outside of this range.

§ **Numerical example,** fig. 27, p.53:
• Potential employee states his "value" as $\widetilde{A} = 1.2$.
• Employer offers $C = 1$.
• Other parameters in figure.
• Increasing solid red curve in fig. 27: $\widehat{h}_{\rm ref}(C,\varepsilon)$.
• Decreasing solid blue curve in fig. 27: $\widehat{h}_{\rm pw}(C,\varepsilon)$.
• Overall robustness, $\widehat{h}(C,PW_{\rm c},\varepsilon) = \min\left[\widehat{h}_{\rm pw}(C,PW_{\rm c}),\ \widehat{h}_{\rm ref}(C,\varepsilon)\right]$, from eq.(260).
• Recall that $\widehat{h}(C,PW_{\rm c},\varepsilon)$ **varies over the plane,** $\varepsilon$ vs $PW_{\rm c}$.
• Suppose $\varepsilon = 0.5$ and $PW_{\rm c} = 1$, then $\widehat{h} = \widehat{h}_{\rm pw} \approx 0.03$ (blue). **Very low robustness.**

§ **Numerical example,** fig. 28, p.53:
• Employer offers lower salary: $C = 0.9$. Other parameters the same.
• $\widehat{h}_{\rm pw}(C,\varepsilon)$ increases: blue solid to green dash. Does this make sense? **Why?**
• $\widehat{h}_{\rm ref}(C,\varepsilon)$ decreases: red solid to turquoise dash. Does this make sense? **Why?**

• Suppose $\varepsilon = 0.5$ and $PW_{\rm c} = 1$, then $\widehat{h} = \widehat{h}_{\rm pw} \approx 0.12$ (dash green). **Better than before. Why?**

Robustness for refusal decreased, but robustness for PW is smaller, and increased more.

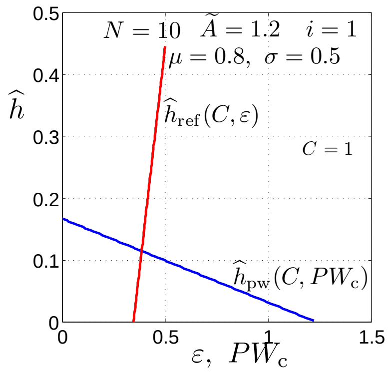


Figure 27: Sub-robustness curves, eqs.(263) (blue) and (269) (red). $C = 1.0$ (Transp.) $i = 0.1$.

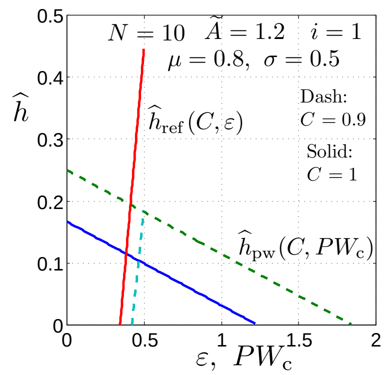


Figure 28: Sub-robustness curves, eqs.(263) (blue, green) and (269) (red, cyan). Solid: $C = 1.0$. Dash: $C = 0.9$ (Transp.). $i = 0.1$.

# 12 Opportuneness: The Other Side of Uncertainty

## 12.1 Opportuneness and Uncertain Constant Yearly Profit, $A$

§ **Return to example in section 8, p.27:**

- **Future worth** of constant profit, eq.(12), p.9:
  - $A$ = profit at end of each period.
  - $i$ = reinvest at profit rate $i$.
  - $N$ = number of periods.
  - The future worth is:

$$FW = \underbrace{\frac{(1+i)^N - 1}{i}}_{\mathcal{I}} A \tag{270}$$

- **Uncertainty:** the constant end-of-period profit, $A$, is uncertain.
  - $\widetilde{A}$ = known estimated profit.
  - $A$ = unknown true profit.
  - $s_A$ = error of estimate.
  - Fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ A : \left| \frac{A - \widetilde{A}}{s_A} \right| \le h \right\}, \quad h \ge 0 \tag{271}$$

- **Robustness:**

$$\widehat{h}(FW_{\rm c}) = \max \left\{ h : \left( \min_{A \in \mathcal{U}(h)} FW(A) \right) \ge FW_{\rm c} \right\} \tag{272}$$

$$= \boxed{\frac{1}{s_A}\left( \widetilde{A} - \frac{FW_{\rm c}}{\mathcal{I}} \right)} \tag{273}$$

§ **Opportuneness:**

- $FW_{\rm w}$ is a wonderful windfall value of $FW$:

$$FW_{\rm w} \ge FW(\widetilde{A}) \ge FW_{\rm c} \tag{274}$$

- Opportuneness:
  - **Uncertainty is good:** The potential for better-than-expected outcome.
  - Distinct from robustness for which **uncertainty is bad.**
  - The investment is **opportune** if $FW_{\rm w}$ is possible at low uncertainty.
  - Investment 1 is **more opportune** than investment 2 if
    $FW_{\rm w}$ is possible at lower uncertainty with investment 1 than with investment 2.
- Definition of opportuneness function:

$$\widehat{\beta}(FW_{\rm w}) = \min \left\{ h : \left( \max_{A \in \mathcal{U}(h)} FW(A) \right) \ge FW_{\rm w} \right\} \tag{275}$$

- Compare with robustness, eq.(272): exchange of min and max operators.
- Meaning of opportuneness function: small $\widehat{\beta}$ is good; large $\widehat{\beta}$ is bad:
  $\widehat{\beta}$ is **immunity against windfall.**
- Meaning of robustness function: small $\widehat{h}$ is bad; large $\widehat{h}$ is good:
  $\widehat{h}$ is **immunity against failure.**

§ **Evaluating the opportuneness.**

- Aspiration exceeds anticipation:

$$FW_{\rm w} > FW(\widetilde{A}) \tag{276}$$

Thus we need favorable surprise to enable $FW_{\rm w}$.

- **Question:** What is opportuneness for $FW_{\rm w} \le FW(\widetilde{A})$?
- $M(h)$ is inner maximum in eq.(275): the inverse of $\widehat{\beta}(FW_{\rm w})$.
- $M(h)$ occurs for $A = \widetilde{A} + s_A h$:

$$M(h) = \mathcal{I}(\widetilde{A} + s_A h) \ge FW_{\rm w} \quad \Longrightarrow \quad \boxed{\widehat{\beta}(FW_{\rm w}) = \frac{1}{s_A}\left(\frac{FW_{\rm w}}{\mathcal{I}} - \widetilde{A}\right)} \tag{277}$$

- **Zeroing:** No uncertainty needed to enable the anticipated value: $FW_{\rm w} = FW(\widetilde{A})$ (fig 29, p.55).
- **Trade off:** Opportuneness gets worse ($\widehat{\beta}$ bigger) as aspiration increases ($FW_{\rm w}$ bigger).

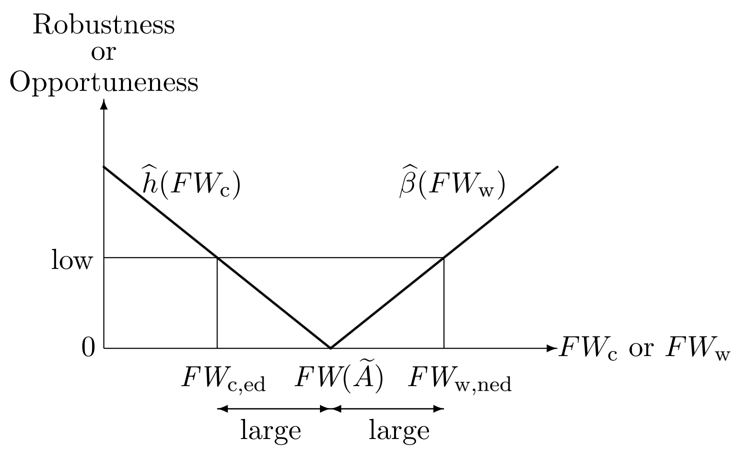


Figure 29: Robustness and opportuneness curves.

§ **Immunity functions: sympathetic or antagonistic:**

- Combine eqs.(273) and (277):

$$\widehat{h} = -\widehat{\beta} + \frac{FW_{\rm w} - FW_{\rm c}}{s_A \mathcal{I}} \tag{278}$$

Note: 2nd term on right is non-negative: $FW_{\rm w} \ge FW_{\rm c}$.

- Robustness and opportuneness are **sympathetic wrt choice of $\widetilde{A}$:**

  Any change in $\widetilde{A}$ that improves robustness also improves opportuneness:

$$\frac{\partial \widehat{h}}{\partial \widetilde{A}} > 0 \quad \text{if and only if} \quad \frac{\partial \widehat{\beta}}{\partial \widetilde{A}} < 0 \tag{279}$$

  Does this make sense? **Why?**

- Robustness and opportuneness are **antagonistic wrt choice of $s_A$:**

  Any change in $s_A$ that improves robustness worsens opportuneness:

$$\frac{\partial \widehat{h}}{\partial s_A} < 0 \quad \text{if and only if} \quad \frac{\partial \widehat{\beta}}{\partial s_A} < 0 \tag{280}$$

  Does this make sense? **Why?**

- Robustness and opportuneness are **sympathetic wrt choice of $x$** if and only if:

$$\frac{\partial \widehat{h}}{\partial x}\frac{\partial \widehat{\beta}}{\partial x} < 0 \tag{281}$$

## 12.2 Robustness and Opportuneness: Sellers and Buyers

§ **Buyers, sellers and diminishing marginal utility (Tal and Gal):**[18]

- **Ed has lots of oranges.** He eats oranges all day. He would **love** an apple.
  Ed's **marginal utility** for oranges is **low** and for apples is **high.**
- **Ned has lots of apples.** He eats apples all day. He would **love** an orange.
  Ned's **marginal utility** for apples is **low** and for oranges is **high.**
- When Ed and Ned meet they rapidly make a deal to exchanges some apples and oranges.

§ This **marginal utility explanation does not explain all transactions,**
especially exchanges of monetary instruments: money for money.

§ **Continue example** in section 12.1, p.54.

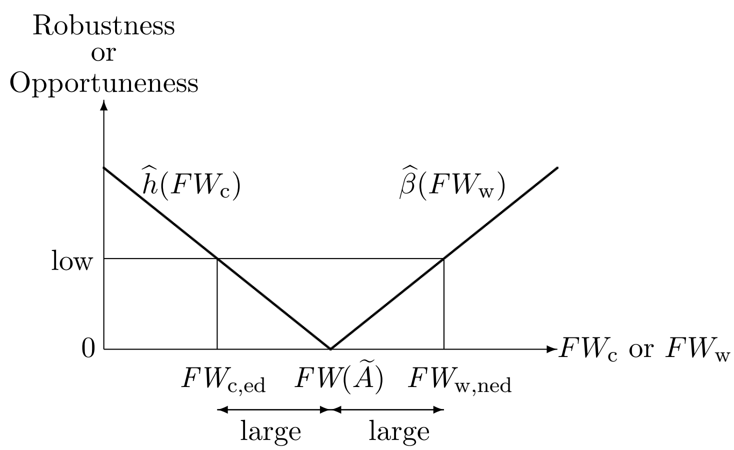


Figure 30: Robustness and opportuneness curves.

§ **Ed wants to own** an investment with **confidence for moderate earnings.**

- Ed's critical $FW$ is $FW_{\rm c,ed}$.
- The robustness, eq.(273), p.54, is (see fig. 30, p.56):

$$\widehat{h}(FW_{\rm c}) = \frac{1}{s_A}\left(\widetilde{A} - \frac{FW_{\rm c}}{\mathcal{I}}\right) \tag{282}$$

- The robustness—immunity against failure—for $FW_{\rm c,ed}$ is low so **Ed wants to sell.** Fig. 30, p.56.

§ **Ned wants to own** an investment with **potential for high earnings.**

- Ned's windfall $FW$ is $FW_{\rm w,ned}$.
- The opportuneness function, eq.(277), p.55, is (see fig. 30, p.56):

$$\widehat{\beta}(FW_{\rm w}) = \frac{1}{s_A}\left(\frac{FW_{\rm w}}{\mathcal{I}} - \widetilde{A}\right) \tag{283}$$

- The opportuneness—immunity against windfall— for $FW_{\rm w,ned}$ is low so **Ed wants to buy.** Fig. 30, p.56.

§ **Ed, meet Ned.** **Ned, meet Ed.** Let's make a deal!

[18] Marginal utility: toelet shulit.

## 12.3 Robustness Indifference and Its Opportuneness Resolution

§ **Continue example of section 12.2,** p.56:

- $A$ = profit at end of each period.
- $i$ = reinvest at profit rate $i$.
- $N$ = number of periods.

§ **The robustness and opportuneness functions are:**

$$\widehat{h}(FW_{\rm c}) = \frac{1}{s_A}\left(\widetilde{A} - \frac{FW_{\rm c}}{\mathcal{I}}\right) \tag{284}$$

$$\widehat{\beta}(FW_{\rm w}) = \frac{1}{s_A}\left(\frac{FW_{\rm w}}{\mathcal{I}} - \widetilde{A}\right) \tag{285}$$

§ **Choice between two plans,** $\widetilde{A}, s_A$ and $\widetilde{A}', s'_A$, where:

$$\widetilde{A} < \widetilde{A}', \quad \frac{\widetilde{A}}{s_A} > \frac{\widetilde{A}'}{s'_A} \tag{286}$$

- The left relation implies that the 'prime' option is purportedly better.
- The right relation implies that the 'prime' option is more uncertain.
- The robustness curves **cross** at $FW_\times$ (see fig. 31, p.57):
  Robust indifference between plans for $FW_{\rm c} \approx FW_\times$.
- The opportuneness curves **do not cross** (see fig. 31):
  Opportuneness preference for plan $\widetilde{A}', s'_A$.
- Opportuneness can resolve a robust indifference.

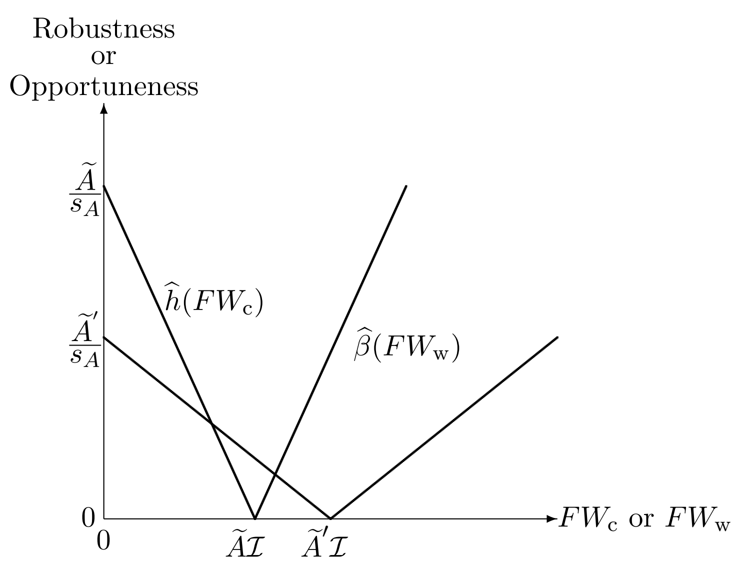


Figure 31: Robustness and opportuneness curves for the two options in eq.(286).