41. **Present worth of yearly profit.** (Based on exam 28.5.2018.) (p.120)

(a) The profit at the end of year n is R_n, where $R_1 = \$2{,}500$, $R_2 = \$3{,}500$, $R_3 = \$5{,}000$. The discount rate is $i = 0.09$. What is the present worth of the total income stream?

(b) The profit at the end of year n is R_n, where $R_1 = \$2{,}500$, $R_2 = \$3{,}500$, $R_3 = \$5{,}000$. At the end of each year you will invest that year's profit, R_n, with yearly rate of return of $i_{\rm a} = 0.15$. What is the total accumulated sum at the end of year 3? What is the present worth of that sum using a discount rate of $i = 0.09$?

(c) The profit at the end of year n is R_n, where the estimated values of these profits are $\widetilde{R}_n$ for $n = 1, 2, 3$. The uncertainty in these estimates is given by this info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R_n - \widetilde{R}_n}{\widetilde{R}_n} \right| \le h, \ \ n = 1, 2, 3 \right\}, \quad h \ge 0 \tag{36}$$

The discount rate is i. You require that the present worth be no less than the critical value $\text{PW}_{\rm c}$. Derive an explicit algebraic expression for the robustness.

(d) The return on an investment is a random variable, R, in the interval $[R_1, \ R_2]$. The investment is a success if the return exceeds the critical value $R_{\rm c}$. The probability of success is:

$$P_{\rm s}(R_{\rm c}) = \begin{cases} 0 & \text{if } R_{\rm c} > R_2 \\ \dfrac{R_2 - R_{\rm c}}{R_2 - R_1}, & \text{if } R_1 \le R_{\rm c} \le R_2 \\ 1 & R_{\rm c} < R_1 \end{cases} \tag{37}$$

However, the value of the critical return, $R_{\rm c}$, is uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ R_{\rm c} : \left| \frac{R_{\rm c} - \widetilde{R}_{\rm c}}{\widetilde{R}_{\rm c}} \right| \le h \right\}, \quad h \ge 0 \tag{38}$$

You require that the probability of success be no less than the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness. Assume that $R_1 \le \widetilde{R}_{\rm c} \le R_2$.

(e) (Variation on part 41a) The profit at the end of year n is R_n, where $R_1 = \$5000$, $R_2 = \$3{,}500$, $R_3 = \$2{,}500$. The discount rate is $i = 0.05$. What is the present worth of the total income stream?

(f) (Variation of part 41b) The profit at the end of year n is R_n, where $R_1 = \$5{,}000$, $R_2 = \$3{,}500$, $R_3 = \$2{,}500$. At the end of each year you will invest that year's profit, R_n, with yearly rate of return of $i_{\rm a} = 0.1$. What is the total accumulated sum at the end of year 3? What is the present worth of that sum using a discount rate of $i = 0.04$?

(g) (Variation on part 41c) The profit at the end of year n is R_n, where the estimated values of these profits are $\widetilde{R}_n$ for $n = 1, 2, 3$. The uncertainty in these estimates is given by this info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R_n - \widetilde{R}_n}{w} \right| \le h, \ \ n = 1, 2, 3 \right\}, \quad h \ge 0 \tag{39}$$

where w is a known positive constant. The discount rate is i. You require that the present worth be no less than the critical value $\text{PW}_{\rm c}$. Derive an explicit algebraic expression for the robustness.

(h) (Variation of part 41d) The return on an investment is a random variable, R, in the interval $[R_1,\ R_2]$. The investment is a success if the return exceeds the critical value $R_{\rm c}$. The probability of success is:

$$P_{\rm s}(R_{\rm c}) = \left\{ \begin{array}{cl} 0 & \text{if } R_{\rm c} > R_2 \\ \dfrac{R_2 - R_{\rm c}}{R_2 - R_1}, & \text{if } R_1 \le R_{\rm c} \le R_2 \\ 1 & R_{\rm c} < R_1 \end{array} \right. \tag{40}$$

However, the value of the critical return, $R_{\rm c}$, is uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ R_{\rm c} : \ \left| \frac{R_{\rm c} - \widetilde{R}_{\rm c}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{41}$$

where w is a known positive constant. You require that the probability of success be no less than the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness. Assume that $R_1 \le \widetilde{R}_{\rm c} \le R_2$.

Solution to problem 41, **Present worth of yearly profit** (p.39).

(41a) The present worth is:

$$\text{PW} = \sum_{n=1}^{N}(1+i)^{-n}R_n = 1.09^{-1}2,500 + 1.09^{-2}3,500 + 1.09^{-3}5,000 = \$9,100.4 \tag{539}$$

(41b) The total accumulated value at the end of year 3 is:

$$V = \sum_{n=1}^{N}(1+i_{\text{a}})^{N-n}R_n = 1.15^2 2,500 + 1.15^1 3,500 + 1.15^0 5,000 = \$12,331 \tag{540}$$

The present worth of this value is:

$$\text{PW}(V) = (1+i)^{-3}V = 1.09^{-3}12,331 = \$9,521.8 \tag{541}$$

(41c) The present worth of the profit stream is:

$$\text{PW}(R) = \sum_{n=1}^{N}(1+i)^{-n}R_n \tag{542}$$

The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R\in\mathcal{U}(h)} \text{PW}(R)\right) \ge \text{PW}_{\text{c}}\right\} \tag{543}$$

Let $m(h)$ denote the inner minimum, which occurs when each profit is as low as possible at horizon of uncertainty h:

$$m(h) = \sum_{n=1}^{N}(1+i)^{-n}(1-h)\widetilde{R}_n = (1-h)\text{PW}(\widetilde{R}) \tag{544}$$

Equating this to the critical value, PW_{c}, and solving for h yields the robustness:

$$\widehat{h} = 1 - \frac{\text{PW}_{\text{c}}}{\text{PW}(\widetilde{R})} \tag{545}$$

or zero if this is negative.

(41d) The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R_{\text{c}}\in\mathcal{U}(h)} P_{\text{s}}(R_{\text{c}})\right) \ge P_{\text{c}}\right\} \tag{546}$$

Let $m(h)$ denote the inner minimum, which occurs when the critical value, R_{c}, is as large as possible at horizon of uncertainty h:

$$m(h) = \begin{cases} \dfrac{R_2 - (1+h)\widetilde{R}_{\text{c}}}{R_2 - R_1} & \text{if } (1+h)\widetilde{R}_{\text{c}} \le R_2 \ \ (\text{equiv: } h \le \dfrac{R_2}{\widetilde{R}_{\text{c}}} - 1) \\ 0 & \text{else} \end{cases} \tag{547}$$

Equating this to the critical value, P_{c}, and solving for h yields the robustness:

$$\widehat{h} = \frac{R_2 - (R_2 - R_1)P_{\text{c}}}{\widetilde{R}_{\text{c}}} - 1 \tag{548}$$

or zero if this is negative. Note that $\widehat{h} \le \dfrac{R_2}{\widetilde{R}_{\text{c}}} - 1$.

(41e) The present worth is:

$$\text{PW} = \sum_{n=1}^{N}(1+i)^{-n}R_n = 1.05^{-1}5,000 + 1.05^{-2}3,500 + 1.05^{-3}2,500 = \$10,096 \quad (549)$$

(41f) The total accumulated value at the end of year 3 is:

$$V = \sum_{n=1}^{N}(1+i_{\text{a}})^{N-n}R_n = 1.1^{2}5,000 + 1.1^{1}3,500 + 1.1^{0}2,500 = \$12,400 \quad (550)$$

The present worth of this value is:

$$\text{PW}(V) = (1+i)^{-3}V = 1.04^{-3}12,400 = \$11,024 \quad (551)$$

(41g) The present worth of the profit stream is:

$$\text{PW}(R) = \sum_{n=1}^{N}(1+i)^{-n}R_n \quad (552)$$

The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R\in\mathcal{U}(h)} \text{PW}(R)\right) \geq \text{PW}_{\text{c}}\right\} \quad (553)$$

Let $m(h)$ denote the inner minimum, which occurs when each profit is as low as possible at horizon of uncertainty h:

$$m(h) = \sum_{n=1}^{N}(1+i)^{-n}\left(\widetilde{R}_n - wh\right) = \text{PW}(\widetilde{R}) - h\text{PW}(w) \quad (554)$$

Equating this to the critical value, PW_{c}, and solving for h yields the robustness:

$$\widehat{h} = \frac{\text{PW}(\widetilde{R}) - \text{PW}_{\text{c}}}{\text{PW}(w)} \quad (555)$$

or zero if this is negative.

(41h) The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R_{\text{c}}\in\mathcal{U}(h)} P_{\text{s}}(R_{\text{c}})\right) \geq P_{\text{c}}\right\} \quad (556)$$

Let $m(h)$ denote the inner minimum, which occurs when the critical value, R_{c}, is as large as possible at horizon of uncertainty h:

$$m(h) = \begin{cases} \dfrac{R_2 - (\widetilde{R}_{\text{c}} + wh)}{R_2 - R_1} & \text{if } \widetilde{R}_{\text{c}} + wh \leq R_2 \ \ (\text{equiv: } h \leq \frac{R_2 - \widetilde{R}_{\text{c}}}{w}) \\ 0 & \text{else} \end{cases} \quad (557)$$

Equating this to the critical value, P_{c}, and solving for h yields the robustness:

$$\widehat{h} = \frac{R_2 - \widetilde{R}_{\text{c}} - (R_2 - R_1)P_{\text{c}}}{w} \quad (558)$$

or zero if this is negative. Note that $\widehat{h} \leq \frac{R_2 - \widetilde{R}_{\text{c}}}{w}$.