50. **Future worth.** (Based on midterm exam, 26.6.2023.) (p.143)

(a) You will invest $P = \$100,000$ now with annual interest rate of $i = 0.15$. What is the future worth after $N = 25$ years?

(b) Consider two alternative investments of the same initial sum, P. The annual interest rates of these two investments are related as:

$$0 < i_1 < i_2 \tag{73}$$

The durations of these investments are related as:

$$N_1 > N_2 > 0 \tag{74}$$

For what values of the ratio $\frac{N_1}{N_2}$ is the first investment preferred, in terms of future worth?

(c) You will invest the sum $\$P$ for N years with annual interest rate i. However, the value of P is uncertain, with estimated value $\widetilde{P}$ and estimated error w, where $w > 0$. The following info-gap model represents the uncertainty in P:

$$\mathcal{U}(h) = \left\{ P : \left| \frac{P - \widetilde{P}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{75}$$

We require that the future worth be no less than $F_{\rm c}$. Derive an explicit algebraic expression for the robustness to uncertainty.

Solution to problem 50 **Future worth.** (Based on midterm exam, 26.6.2023.) (p.50).

50a. Future and present worth are related as:

$$F = (1+i)^N P = \$1.15^{25} \times 100,000 = \$3.2929 \times 10^6 \tag{712}$$

50b. The future worths of the two alternatives are:

$$F_1 = (1+i_1)^{N_1} P, \quad F_2 = (1+i_2)^{N_2} P \tag{713}$$

Preference for the 1st investment is expressed as follows:

$$\frac{F_1}{F_2} = \frac{(1+i_1)^{N_1}}{(1+i_2)^{N_2}} > 1 \quad \Longleftrightarrow \quad (1+i_1)^{N_1} > (1+i_2)^{N_2} \tag{714}$$

$$\Longleftrightarrow \quad N_1 \ln(1+i_1) > N_2 \ln(1+i_2) \tag{715}$$

$$\Longleftrightarrow \quad \frac{N_1}{N_2} > \frac{\ln(1+i_2)}{\ln(1+i_1)} \tag{716}$$

We prefer the 1st investment for all values of $\frac{N_1}{N_2}$ satisfying eq.(716).

50c. The definition of the robustness function is:

$$\widehat{h}(F_{\rm c}) = \max\left\{ h: \ \left(\min_{P\in\mathcal{U}(h)} F \right) \ge F_{\rm c} \right\} \tag{717}$$

where $F = (1+i)^N P$.

Let $m(h)$ denote the inner minimum in eq.(717). $m(h)$ occurs for $P = \widetilde{P} - wh$ so:

$$m(h) = (1+i)^N (\widetilde{P} - wh) \ge F_{\rm c} \quad \Longrightarrow \quad \widehat{h}(F_{\rm c}) = \frac{(1+i)^N \widetilde{P} - F_{\rm c}}{(1+i)^N w} = \frac{F(\widetilde{P}) - F_{\rm c}}{(1+i)^N w} \tag{718}$$

or zero if this is negative.