52. **Future Worth and More: 1.** (Based on final exam, 19.7.2023). (p.146). The following questions are independent of one another.

(a) Jane will invest $\$F$ at the start of each year for $N = 25$ years at annual interest of $i = 0.05$. The future worth of this investment is $\$10^5$. What is the value of F?

(b) The future worth of an investment of $\$A$ now is cA where c is an uncertain constant whose estimated value is $\widetilde{c}$. An info-gap model for uncertainty in c is:

$$\mathcal{U}(h) = \left\{ c : \left| \frac{c - \widetilde{c}}{w} \right| \leq h \right\}, \quad h \geq 0 \tag{82}$$

where w is a known positive constant. We require that the future worth be no less than the critical value FW_c. Derive an explicit algebraic expression for the robustness to uncertainty.

(c) Consider two different development projects whose estimated future worths are $\widetilde{FW}_1$ and $\widetilde{FW}_2$. However, these future worths are uncertain. The robustness to uncertainty of the future worth of project k, for $k = 1$ or 2, is:

$$\widehat{h}_k(\text{FW}_\text{c}) = \frac{\widetilde{FW}_k - \text{FW}_\text{c}}{s_k} \tag{83}$$

or zero if this expression is negative. s_k is a known positive value. The following two relations hold:

$$s_1 < s_2, \quad \widetilde{FW}_1 < \widetilde{FW}_2 \tag{84}$$

Sketch the robustness curves and derive an algebraic expression for the range of values of the critical future worth, FW_c, for which project 1 robust-preferred.

(d) Consider N products and services whose prices in year k were $p_{1k}, \ldots, p_{Nk}$. The consumer price index for year k is defined as:

$$I_k = \sum_{j=1}^{N} w_j p_{jk} \tag{85}$$

where $w_1, \ldots, w_N$ are non-negative normalized weights. The inflation with respect to year $k-1$ is defined as:

$$f_k = \frac{I_k - I_{k-1}}{I_{k-1}} \tag{86}$$

The prices in year $k-1$ are known accurately, however the prices in year k are uncertain, where the estimated prices are $\widetilde{p}_{1k}, \ldots, \widetilde{p}_{Nk}$. The price uncertainty is represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ p_{1k}, \ldots, p_{Nk} : \left| \frac{p_{jk} - \widetilde{p}_{jk}}{\widetilde{p}_{jk}} \right| \leq h, \ j = 1, \ldots, N \right\}, \quad h \geq 0 \tag{87}$$

We require that the inflation in year k not exceed the critical value f_c:

$$f_k \leq f_\text{c} \tag{88}$$

Derive an explicit algebraic expression for the robustness to uncertainty.

(e) Consider a project for which the initial investment is S and the real earnings at the end of each year, for N years, will be $R = \frac{S}{N}$. The annual inflation, f, is less than the nominal interest rate i_nom. Is this project economically justified? Explain your answer.

Solution to problem 52 **Future Worth and More: 1.** (p.52).

52a. The future worth at the end of N years is:

$$\text{FW} = F\sum_{k=1}^{N}(1+i)^k = F\frac{(1+i)^{N+1}-(1+i)}{i} = \frac{1.05^{26}-1.05}{0.05}F = 50.1135F = 10^5 \tag{735}$$

Thus:

$$\boxed{F = \$1995.47} \tag{736}$$

52b. The definition of the robustness is:

$$\widehat{h}(\text{FW}_\text{c}) = \max\left\{h:\ \left(\min_{c\in\mathcal{U}(h)} F\right) \geq \text{FW}_\text{c}\right\} \tag{737}$$

Let $m(h)$ denote the inner minimum, which occurs at $c = \widetilde{c} - wh$. Thus:

$$(\widetilde{c} - wh)A \geq \text{FW}_\text{c} \quad\Longrightarrow\quad \boxed{\widehat{h}(\text{FW}_\text{c}) = \frac{1}{w}\left(\widetilde{c} - \frac{\text{FW}_\text{c}}{A}\right)} \tag{738}$$

or zero if this is negative.

52c. The sketch of the robustness curves is in fig. 16. Project 1 is robust-preferred for $\text{FW}_\text{c} \leq \text{FW}_\times$ where:

$$\widehat{h}_1(\text{FW}_\text{c}) = \widehat{h}_2(\text{FW}_\text{c}) \implies \frac{\widetilde{FW}_1 - \text{FW}_\times}{s_1} = \frac{\widetilde{FW}_2 - \text{FW}_\times}{s_2} \tag{739}$$

which implies:

$$\frac{\widetilde{FW}_1}{s_1} - \frac{\widetilde{FW}_2}{s_2} = \left(\frac{1}{s_1} - \frac{1}{s_2}\right)\text{FW}_\times \implies \text{FW}_\times = \frac{\widetilde{FW}_1 s_2 - \widetilde{FW}_2 s_1}{s_2 - s_1} \tag{740}$$

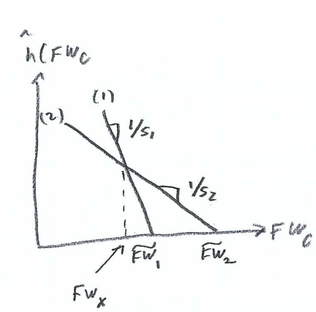

Figure 16: Fig. for solution of problem 52c.

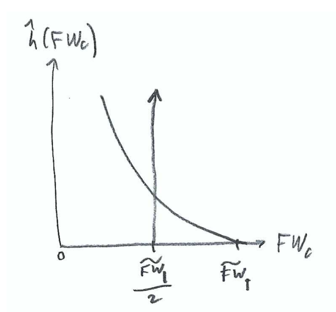

Figure 17: Fig. for solution of problem 53c.

52d. The definition of the robustness is:

$$\widehat{h}(f_\text{c}) = \max\left\{h:\ \left(\max_{p_{jk}\in\mathcal{U}(h)} f\right) \leq f_\text{c}\right\} \tag{741}$$

Let $m(h)$ denote the inner maximum, which occurs when the p_{jk} are maximal:

$$m(h) = \frac{\sum_{j=1}^{N} w_j p_{jk}}{\sum_{j=1}^{N} w_j p_{jk-1}} - 1 = \frac{(1+h)\sum_{j=1}^{N} w_j \widetilde{p}_{jk}}{\sum_{j=1}^{N} w_j p_{jk-1}} - 1 = \frac{(1+h)\widetilde{I}_k}{I_{k-1}} - 1 \leq f_\text{c} \tag{742}$$

Solving for h at equality yields the robustness function:

$$(1+h)\widetilde{I}_k = (1+f_\text{c})I_{k-1} \quad\Longrightarrow\quad \boxed{\widehat{h}(f_\text{c}) = (1+f_\text{c})\frac{I_{k-1}}{\widetilde{I}_k} - 1} \tag{743}$$

or zero if this is negative.

52e. The initial investment, at time $t = 0$, is S and the real annual income is $R = \frac{S}{N}$ at the end of each of N years. It is true that $S = NR$. However, the present worth of N increments of future income, with positive inflation less than the nominal interest, will be less than the initial investment. Clearly this project is not economically justified.

We can see this more explicitly with the expression for present worth, though this is not necessary to answer the question of whether or this project is economically justified. There is no need to actually calculate the numerical value of the present worth: it will be negative.

The real interest rate is:

$$i_{\rm r} = \frac{i_{\rm nom} - f}{1 + f} \tag{744}$$

which is positive because $f < i_{\rm nom}$.

The present worth of this project is:

$$\text{PW} = -S + \sum_{n=1}^{N} (1 + i_{\rm r})^{-k} R \tag{745}$$

The N terms $(1 + i_{\rm r})^{-k}$ are all less than 1, so the righthand side of eq.(745) is negative: the project is not economically justified.